

THE QUENCHED ASYMPTOTICS FOR NONLOCAL SCHRÖDINGER OPERATORS WITH POISSONIAN POTENTIALS

KAMIL KALETA AND KATARZYNA PIETRUSKA-PALUBA

ABSTRACT. We study the quenched long time behaviour of the survival probability up to time t , $\mathbf{E}_x[e^{-\int_0^t V^\omega(X_s)ds}]$, of a symmetric Lévy process with jumps, under a sufficiently regular Poissonian random potential V^ω on \mathbb{R}^d . Such a function is a probabilistic solution to the parabolic equation involving the nonlocal Schrödinger operator based on the generator of the process $(X_t)_{t \geq 0}$ with potential V^ω . For a large class of processes and potentials of finite range, we determine rate functions $\eta(t)$ and compute explicitly the positive constants C_1, C_2 such that

$$-C_1 \leq \liminf_{t \rightarrow \infty} \frac{\log \mathbf{E}_x[e^{-\int_0^t V^\omega(X_s)ds}]}{\eta(t)} \leq \limsup_{t \rightarrow \infty} \frac{\log \mathbf{E}_x[e^{-\int_0^t V^\omega(X_s)ds}]}{\eta(t)} \leq -C_2,$$

almost surely with respect to ω , for every fixed $x \in \mathbb{R}^d$. The functions $\eta(t)$ and the bounds C_1, C_2 heavily depend on the intensity of large jumps of the process. In particular, if its decay at infinity is ‘sufficiently fast’, then we prove that $C_1 = C_2$, i.e. the limit exists. Representative examples in this class are relativistic stable processes with Lévy-Khintchine exponents $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$, $\alpha \in (0, 2)$, $m > 0$, for which we obtain that

$$\lim_{t \rightarrow \infty} \frac{\log \mathbf{E}_x[e^{-\int_0^t V^\omega(X_s)ds}]}{t/(\log t)^{2/d}} = \frac{\alpha}{2} m^{1-\frac{2}{\alpha}} \left(\frac{\rho \omega_d}{d} \right)^{\frac{d}{2}} \lambda_1^{BM}(B(0, 1)), \quad \text{for almost all } \omega,$$

where $\lambda_1^{BM}(B(0, 1))$ is the principal eigenvalue of the Brownian motion killed on leaving the unit ball, ω_d is the Lebesgue measure of a unit ball and $\rho > 0$ corresponds to V^ω . We also identify two interesting regime changes (‘transitions’) in the growth properties of the rates $\eta(t)$ as the intensity of large jumps of the processes varies from polynomial to higher order, and eventually to stretched exponential order.

Key-words and phrases: symmetric Lévy process, random nonlocal Schrödinger operator, parabolic nonlocal Anderson model, Feynman-Kac semigroup, random Poissonian potential, principal (ground state) eigenvalue, integrated density of states, annealed asymptotics, quenched asymptotics, relativistic process

2010 *MS Classification:* Primary 60G51, 60H25, 60K37; Secondary 47D08, 60G60, 47G30

1. Introduction

This paper is concerned with the large time asymptotic behaviour of the solutions of the spatially continuous parabolic nonlocal Anderson problem with Poissonian interaction, driven by a Lévy process in \mathbb{R}^d . More precisely, we consider the equation

$$(1.1) \quad \partial_t u = Lu - V^\omega u, \quad u(0, x) \equiv 1,$$

where L is the generator of the underlying process and $V^\omega(x) = \int_{\mathbb{R}^d} W(x-y) \mu^\omega(dy)$ is a random Poissonian potential with sufficiently regular profile function $W : \mathbb{R}^d \rightarrow \mathbb{R}_+$. By μ^ω we denote the Poisson random measure on \mathbb{R}^d with intensity ρdx , $\rho > 0$, over a given probability space (Ω, \mathbb{Q}) .

Processes considered throughout the paper, $X = (X_t, \mathbf{P}_x)_{t \geq 0, x \in \mathbb{R}^d}$, are symmetric Lévy processes with jumps, with characteristic functions

$$\mathbf{E}_0 \left[e^{i\xi \cdot X_t} \right] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, \quad t > 0,$$

⁰ Research supported by the National Science Centre (Poland) internship grant on the basis of the decision No. DEC-2012/04/S/ST1/00093 and by the Foundation for Polish Science.

whose characteristic exponents (symbols) ψ are given by the Lévy-Khintchine formula

$$(1.2) \quad \psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot z)) \nu(dz).$$

Here $A = (a_{ij})_{1 \leq i, j \leq d}$ is a symmetric non-negative definite matrix, and ν is a symmetric Lévy measure, i.e. a Radon measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(dz) < \infty$ and $\nu(E) = \nu(-E)$, for every Borel $E \subset \mathbb{R}^d \setminus \{0\}$ [15, 16]. We always assume that X is strong Feller and $e^{-t_0 \psi(\cdot)} \in L^1(\mathbb{R}^d)$, for some $t_0 > 0$ (for more details see Section 2.1).

Since its introduction in the 50's of the past century, the parabolic Anderson model based on the Laplacian (both continuous and discrete), with various potentials, has been studied with varying intensity. For an excellent review of the history of the research in this area we refer to the book of König [25].

Under suitable regularity assumptions, the solution to the problem (1.1) can be probabilistically represented by means of the Feynman-Kac formula:

$$(1.3) \quad u^\omega(t, x) = \mathbf{E}_x \left[e^{-\int_0^t V^\omega(X_s) ds} \right].$$

One is interested in the long-time behaviour of $u^\omega(t, x)$, in both the annealed sense (averaged with respect to \mathbb{Q}) and the quenched sense (almost sure with respect to \mathbb{Q}). In this paper, we will analyse the quenched behaviour of functionals $u^\omega(t, x)$ for Lévy processes whose exponent ψ can be written as

$$(1.4) \quad \psi(x) = \psi^{(\alpha)}(x) + o(|x|^\alpha), \quad |x| \rightarrow 0,$$

for some $\alpha \in (0, 2]$, and satisfies some mild assumptions concerning its behaviour at infinity. In formula (1.4), $\psi^{(\alpha)}$ is the characteristic exponent of a symmetric (not necessarily isotropic) α -stable process, i.e. a Lévy process with characteristic exponent

$$(1.5) \quad \psi^{(\alpha)}(\xi) = \int_0^\infty \int_{S^{d-1}} \frac{1 - \cos(\xi \cdot rz)}{r^{1+\alpha}} n(dz) dr,$$

where n is a symmetric finite measure on unit sphere S^{d-1} when $\alpha \in (0, 2)$, or

$$(1.6) \quad \psi^{(\alpha)}(\xi) = \xi \cdot A\xi,$$

where $A = (a_{ij})_{1 \leq i, j \leq d}$ is a symmetric nonnegative definite matrix when $\alpha = 2$. When n is the uniform distribution on S^{d-1} for $\alpha \in (0, 2)$ or $A \equiv a \text{Id}$ with some $a > 0$ for $\alpha = 2$, then the process is called *isotropic α -stable*. We assume the nondegeneracy condition $\inf_{|\xi|=1} \psi^{(\alpha)}(\xi) > 0$.

The annealed asymptotics of $u^\omega(t, x)$ has been first analyzed by Donsker-Varadhan [10] (for stable processes, including the Brownian motion) and of Okura [27] (for symmetric Lévy processes satisfying (1.4)). When the profile W is of order $o(1/|x|^{d+\alpha})$ when $|x| \rightarrow \infty$, they prove that

$$(1.7) \quad \lim_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} [u^\omega(t, x)]}{t^{d/(d+\alpha)}} = -(\rho \omega_d)^{\frac{\alpha}{d+\alpha}} \left(\frac{d+\alpha}{\alpha} \right) \left(\frac{2\lambda_{(\alpha)}}{d} \right)^{\frac{d}{d+\alpha}},$$

In this formula, ω_d is the volume of the unit ball, and

$$\lambda_{(\alpha)} = \inf_{U \text{ open}, |U|=\omega_d} \lambda_1^{(\alpha)}(U)$$

denotes the infimum of principal eigenvalues for the symmetric α -stable process with exponent (1.5) in U with outer Dirichlet conditions on U^c . Okura's work covers also the case when $\psi(x) = O(\psi^{(\alpha)}(x))$, $|x| \rightarrow 0$, but only when the potential is heavy-tailed. This falls not within the scope of present paper and so we will discuss this case elsewhere.

The key observation used in the quenched case is that when the profile function W is of bounded support, then \mathbb{Q} -a.s. there exist large areas with no potential interaction. Typically, with high probability, the process

tends to remain in those ‘atypical’, ‘favorable’ areas, which affects the a.s. behaviour of the functional. As a result, the quenched behaviour can differ from the annealed asymptotics.

This phenomenon (for the Brownian motion only) was first observed and rigorously established by Sznitman in [32]. He proves that in that case, for any $x \in \mathbb{R}^d$, and \mathbb{Q} –almost all ω ,

$$(1.8) \quad \lim_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}} = - \left(\frac{\rho \omega_d}{d} \right)^{\frac{2}{d}} \lambda_1^{BM}(B(0, 1)),$$

where $\lambda_1^{BM}(B(0, 1))$ is the principal eigenvalue for the Brownian motion killed on exiting $B(0, 1)$. This result was reproven by Fukushima [11]. For the Brownian motion on some irregular spaces such as the Sierpiński gasket, one also sees a similar phenomenon: rates of the annealed and the quenched asymptotics differ (see [28, 29]).

In this paper, we address the quenched asymptotics for Lévy processes with jumps influenced by potentials with compact-range profiles. Key examples include a vast selection of isotropic unimodal Lévy processes, subordinate Brownian motions, processes with nondegenerate Brownian components and with non-isotropic Lévy densities as well as processes with less regular Lévy measures that have product or discrete large jumps components. While the ‘favorable’ spots in the Poissonian configuration are still present, the jumping nature of Lévy processes drives the process out of those spots: if the process does not stay there long enough, then the effect of ‘no-potential-interaction’ is spoiled and as a consequence the quenched rate can be the same as the annealed rate. What is decisive here is the intensity of long jumps of the process: for processes with Lévy measures whose tails decay fast enough at infinity, we see the same phenomenon as that for Brownian motion.

For more clarity, we have collected the results obtained for particular classes of processes with various types of large jump intensities in Table 1 below (for simplicity we restricted the presentation to the family of isotropic unimodal Lévy processes with stable-type small jumps).

The annealed rate is always governed by the exponent α appearing in (1.4), which is determined by the behaviour of the exponent of ψ near zero. Formula (1.4) together with some mild assumptions concerning the behaviour of the symbol at infinity permit to obtain the annealed asymptotics of $u^\omega(t, x)$ and also to identify the constant in (1.7).

The question of the quenched rate is much more delicate. In this case, the formula (1.4) (even if combined with some information on the behaviour of the characteristic exponent at infinity) is generally insufficient. This is particularly evident when $\alpha = 2$. It occurred to us as a surprise that the effective derivation of the quenched rate (and the corresponding bounds) requires deep analysis of the subtle properties of Lévy processes with prescribed Lévy measures, depending on the type of their fall-off at infinity.

As usual, in this paper the upper and the lower bounds of $u^\omega(t, x)$ are addressed separately. First, in Sections 3 and 4 we prove two general results: Theorem 3.1 concerning the upper bound, and Theorem 4.1 concerning the lower bound.

The rest of the paper (Section 5) is devoted to the application of our general results for specific classes of processes.

(1) For processes satisfying (1.4) with $\alpha \in (0, 2)$ (Theorem 5.2 and Examples 5.1-5.2), and also for those with $\alpha = 2$ but polynomially decaying Lévy measures (Theorem 5.3 and Examples 5.3, 5.5 (2)), the quenched and annealed rates coincide and are both equal to $t^{d/(d+\alpha)}$.

(2) If the Lévy measure decays stretched exponentially or faster (one must necessarily have $\alpha = 2$ in this case), then the annealed rate is $t^{d/(d+2)}$, while the quenched rate is bigger and equal to $t/(\log t)^{2/d}$. This is the same rate as that for the Brownian motion. In this case, we not only identify the quenched rate, but also often obtain the limit (Theorem 5.5 and Examples 5.4, 5.5 (1)). This case covers many examples of

intensity of jumps / process	parameters	rate $\eta(t)$	lower bound for $\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{\eta(t)}$	upper bound for $\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{\eta(t)}$	limit
$\frac{C}{r^{d+\alpha}}$	$\alpha \in (0, 2)$	$t^{\frac{d}{d+\alpha}}$	$-\frac{4\alpha+9d}{2} \left(\frac{2}{d+2\alpha}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_{1,\nu}^{(\alpha)}\right)^{\frac{d}{d+\alpha}}$	$-\alpha \left(\frac{2}{d+2\alpha}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_{1,\nu}^{(\alpha)}\right)^{\frac{d}{d+\alpha}}$	no
$\frac{C}{r^{d+\alpha}} \mathbf{1}_{\{r \leq 1\}} + \frac{C}{r^{d+\delta}} \mathbf{1}_{\{r > 1\}}$	$\alpha \in (0, 2)$ $\delta > 2$	$t^{\frac{d}{d+2}}$	$-\frac{4\delta+9d}{2} \left(\frac{2}{d+2\delta}\right)^{\frac{d}{d+2}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d+2}} \left(\lambda_{1,\nu}^{(2)}\right)^{\frac{d}{d+2}}$	$-\delta \left(\frac{2}{d+2\delta}\right)^{\frac{d}{d+2}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d+2}} \left(\lambda_{1,\nu}^{(2)}\right)^{\frac{d}{d+2}}$	no
$\frac{C}{r^{d+\alpha}} \mathbf{1}_{\{r \leq 1\}} + \frac{C}{e^{\theta(\log r)^\beta}} \mathbf{1}_{\{r > 1\}}$	$\alpha \in (0, 2)$ $\theta > 0$ $\beta > 1$	$t^{\frac{d\beta}{d\beta+2}}$	$-2\theta^{\frac{2}{2+d\beta}} \left(\frac{\omega_d\rho}{d}\right)^{\frac{2\beta}{2+d\beta}} \left(\lambda_{1,\nu}^{(2)}\right)^{\frac{d\beta}{2+d\beta}}$	$-\theta^{\frac{2}{2+d\beta}} \left(\frac{\omega_d\rho}{d}\right)^{\frac{2\beta}{2+d\beta}} \left(\lambda_{1,\nu}^{(2)}\right)^{\frac{d\beta}{2+d\beta}}$	no
$\frac{C}{r^{d+\alpha}} \mathbf{1}_{\{r \leq 1\}} + \frac{C}{e^{\theta(r-1)^\beta}} \mathbf{1}_{\{r > 1\}}$	$\alpha \in (0, 2)$ $\theta > 0$ $\beta \in (0, 1)$	$\frac{t}{(\log t)^{2/d}}$	$-\beta^{\frac{2}{d}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	$-\beta^{\frac{2}{d}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	yes
$\frac{C}{r^{d+\alpha}} \mathbf{1}_{\{r \leq 1\}} + \frac{C}{e^{\theta(r-1)^\beta}} \mathbf{1}_{\{r > 1\}}$	$\alpha \in (0, 2)$ $\theta > 0$ $\beta \geq 1$	$\frac{t}{(\log t)^{2/d}}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	yes
$\frac{C}{r^{d+\alpha}} \mathbf{1}_{\{r \leq 1\}}$	$\alpha \in (0, 2)$	$\frac{t}{(\log t)^{2/d}}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_{1,\nu}^{(2)}$	yes
Brownian motion		$\frac{t}{(\log t)^{2/d}}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_1^{BM}$	$-\left(\frac{\rho\omega_d}{d}\right)^{\frac{2}{d}} \lambda_1^{BM}$	yes

TABLE 1. Rate functions $\eta(t)$ and bounds for $\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{\eta(t)}$ and $\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{\eta(t)}$ for specific isotropic Lévy processes. First six examples are pure jump processes with Lévy-Khintchine exponents as in (1.2) with $A \equiv 0$ and $\nu(dx) = \nu(|x|)dx$, where $\nu(r)$ are subsequent profiles given in the first column. Here $\lambda_{1,\nu}^{(\alpha)}$ and $\lambda_{1,\nu}^{(2)}$ denote the principal eigenvalues for the given stable process (in the first line) and diffusions determined by Gaussian matrices as in (5.20) (in the next five lines), killed on leaving the ball $B(0, 1)$. We compare these examples with the case of Brownian motion which is included in the last line. In the last column we indicate for which processes the convergence follows.

processes that are of interests in mathematical physics and in technical sciences, including the relativistic α -stable process and some tempered stable processes [6, 18].

(3) We also consider a class of processes with Lévy measures that have intermediate decay: slower than stretched exponential, but still faster than polynomial (Theorem 5.4). The annealed rate is *perforce* equal to $t^{d/(d+2)}$, but the quenched rate obtained is $t^{(\beta d)/(2+\beta d)}$, β being a parameter specific to the process.

It is seen from this picture that the two interesting regime changes ('transitions') in the growth properties of the quenched rates occur. The first one can be observed when the intensity of large jumps of the processes varies from polynomial to higher order, in the sense that the quenched rate becomes faster than the annealed rate (i.e. it is no longer consistent with the annealed rate and becomes heavily dependent on the decay of the intensity of large jumps of the process). The second transition occurs as the intensity of large jumps becomes stretched exponential or faster. In this case, the long jumps intensity-driven quenched rate takes the form $t/(\log t)^{2/d}$, which is the quickest possible one, obtained also for Brownian motion. It is worth to point out that similar large jumps intensity-dependent transition in the ground state fall-off properties of the nonlocal Schrödinger operators has been recently identified in [19]

The verification of the assumptions of our general Theorems 3.1 and 4.1 for various types of Lévy measures (i.e. in each of the situations (1)-(3) above) requires a separate analysis. The applicability of our results essentially depends on the verifiability of the assumption (U) preceding Theorem 3.1. It asserts the existence of the profile function $F(r)$ that dominates the tail $\mathbf{P}_0(|X_t| > r)$ for large r . This profile plays a crucial role in determining the quenched rate and therefore, in applications, it is a key initial step to establish it as precisely as possible. It does not come as a surprise that such a profile should be determined by the tail of the corresponding Lévy measure. When (1.4) holds with $\alpha \in (0, 2)$, then the corresponding profiles $F(r)$ are derived by using the general estimates for the tails of the supremum functional obtained in [30]. When $\alpha = 2$, the problem is more complicated and it requires an application of the sharp estimates of the transition probability densities that are available in the literature. For Lévy measures with stretched exponential and lighter tails, we apply directly the results of [7] while for those with polynomial and intermediate tails we use the estimates obtained recently in [21] (Lemmas 5.2-5.3). The case of jump processes with non-degenerate Gaussian components is discussed separately in Proposition 5.3. Another key step in application of our general lower bound was to find a possibly sharpest lower estimate for the Dirichlet heat kernels of the large box which leads to sufficiently precise lower bound of the function G defined in (4.4). For processes with Lévy measures whose tails decay at infinity not faster than exponentially this is established in Proposition 5.4. The cases with lighter tails require an application of more specialized estimates obtained in [24].

At the end of the Introduction, let's say a few words about how the general theorems Th. 3.1, Th. 4.1 are obtained. To the best of our knowledge the quenched asymptotics for Lévy processes with jumps has not been studied before. In the literature concerning the Brownian motion, one finds two methods: Sznitman's paper [32] estimates $u^\omega(t, x)$ directly, using his 'enlargement of obstacles' technique for the more difficult upper bound (similar method was used on the Sierpiński gasket in [28]); Fukushima [11] gives elegant arguments for deriving both the upper and the lower quenched bound from respective upper and lower bounds at zero for the integrated density of states of the corresponding Schrödinger operator (being closely related to the annealed upper and lower bounds) - this is done by means of the Dirichlet-Neumann bracketing for the Laplace operator. In our work, we are able to find a counterpart of Fukushima's method for Lévy processes with jumps to obtain the upper bounds. As the Dirichlet-Neumann bracketing seems not to be available in the nonlocal case, we had to use a different approach for the lower bound. The lower estimate of $u^\omega(t, x)$ we provide is proven directly, without using any properties of the annealed limits.

2. Preliminaries

2.1. Lévy processes

Recall that $X = (X_t)_{t \geq 0}$ is assumed to be a symmetric jump Lévy process in \mathbb{R}^d , $d \geq 1$, with Lévy-Khintchine exponent ψ as in (1.2). We will always assume that X is strong Feller and

$$(2.1) \quad \text{there exists } t_0 > 0 \text{ for which } e^{-t_0 \psi(\cdot)} \in L^1(\mathbb{R}^d).$$

Note that the strong Feller property is equivalent to the existence of measurable transition densities $p(t, x, y) = p(t, y - x)$ for the process (see e.g. [31, Th. 27.7]), while (2.1) guarantees that $\sup_{x \in \mathbb{R}^d} p(t, x) = p(t, 0) \leq p(t_0, 0) < \infty$, for any $t \geq t_0$.

Consequently, X is strong Markov with respect to its natural filtration and has a modification with càdlàg paths. The càdlàg property will be assumed throughout the paper. For more details on Lévy processes we refer to [31, 15, 16, 1].

The generator L of the process $(X_t)_{t \geq 0}$ is a nonlocal pseudodifferential operator uniquely determined by its Fourier transform

$$(2.2) \quad \widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^d, \quad f \in \mathcal{D}(L),$$

where $\mathcal{D}(L) = \{f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d)\}$. It is a negative-definite self-adjoint operator with a core $C_0^\infty(\mathbb{R}^d)$ such that for $f \in C_0^\infty(\mathbb{R}^d)$

$$Lf(x) = \sum_{i,j=1}^d a_{ij} \frac{\partial^2 f}{\partial x_j \partial x_i}(x) + \int_{\mathbb{R}^d \setminus \{0\}} (f(x+z) - f(x) - z \cdot \nabla f(x) 1_{\{|z| \leq 1\}}(z)) \nu(dz), \quad x \in \mathbb{R}^d.$$

The corresponding Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ can be defined by

$$(2.3) \quad \mathcal{E}(f, g) = \int_{\mathbb{R}^d} \psi(\xi) \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi, \quad f, g \in D(\mathcal{E}),$$

with $D(\mathcal{E}) = \{f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \psi(\xi) |\widehat{f}(\xi)|^2 d\xi < \infty\}$. It holds that $\mathcal{E}(f, g) = (-Lf, g)$, for $f \in \mathcal{D}(L)$ and $g \in \mathcal{D}(\mathcal{E})$.

The transition densities $p^U(t, x, y)$ of the process killed upon exiting an open, bounded set $U \subset \mathbb{R}^d$ are given by the Dynkin-Hunt formula

$$(2.4) \quad p^U(t, x, y) = p(t, x, y) - \mathbf{E}_x[\tau_U < t; p(t - \tau_U, X_{\tau_U}, y)], \quad x, y \in U.$$

Here and thereafter, $\tau_U = \inf\{t \geq 0 : X_t \notin U\}$ denotes the exit time of the process from the set U . The L^2 -semigroup of operators with kernel $p^U(\cdot, \cdot, \cdot)$, also called the Dirichlet semigroup, will be denoted by $\{P_t^U : t \geq 0\}$. Since U is bounded, the operators P_t^U are trace-class (consequently, compact) and admit a complete set of positive eigenvalues

$$\lambda_1(U) < \lambda_2(U) \leq \lambda_3(U) \dots \rightarrow \infty.$$

Sometimes, to specify which process we are working with, these eigenvalues will be denoted by $\lambda_i^\psi(U)$, where ψ is the Lévy exponent of $(X_t)_{t \geq 0}$. In the special case of symmetric α -stable processes, $\alpha \in (0, 2]$, its corresponding Dirichlet form will be denoted by $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$, and the eigenvalues of the Dirichlet semigroup – by $\lambda_i^{(\alpha)}(U)$. For the standard Brownian motion running at twice the speed, we will use the notation $(\mathcal{E}^{BM}, \mathcal{D}(\mathcal{E}^{BM}))$ and $\lambda_i^{BM}(U)$, respectively.

2.2. Poisson potentials

The process X will be subject to interaction with a nonnegative, random Poissonian potential V^ω . To properly set the assumptions, recall that the Kato class relative to X , \mathcal{K}^X , consists of those measurable functions $V : \mathbb{R}^d \rightarrow \mathbb{R}$ for which

$$(2.5) \quad \lim_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \mathbf{E}_x |V(X_s)| \, ds = 0,$$

and the local Kato class $\mathcal{K}_{\text{loc}}^X$ – of functions V such that for every ball $B = B(x, r)$ the function $V \cdot \mathbf{1}_B \in \mathcal{K}^X$. We always have $L_{\text{loc}}^\infty(\mathbb{R}^d) \subset \mathcal{K}_{\text{loc}}^X \subset L_{\text{loc}}^1(\mathbb{R}^d)$. The condition defining the Kato class can be reformulated in terms of the kernel $p(t, x)$ restricted to small t and small x : it is shown in [13, Corollary 1.3] that (2.5) is equivalent to

$$(2.6) \quad \lim_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{B(x, t)} p(s, x - y) |V(y)| \, dy \, ds = 0.$$

Sharp estimates of $p(t, x)$ that are available in the literature (see e.g. [21, 20, 2, 7]) often allow to find more explicit form of (2.6).

Further, let \mathcal{N} be a Poisson point process on \mathbb{R}^d , with intensity $\rho \, dx$, $\rho > 0$, defined on some probability space $(\Omega, \mathcal{M}, \mathbb{Q})$, and let $W : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a $\mathcal{K}_{\text{loc}}^X$ – function satisfying

$$(2.7) \quad \sup_{x \in B(0, R)} W(x - \cdot) \in L^1(B(0, 2R)^c), \quad \text{for every } R > 1.$$

Then define

$$(2.8) \quad V^\omega(x) = \int_{\mathbb{R}^d} W(x - y) \mu^\omega(dy),$$

where μ^ω is the random counting measure on \mathbb{R}^d corresponding to the Poisson point process \mathcal{N} . For such profiles W , the potential $V^\omega(\cdot)$ belongs \mathbb{Q} –almost surely to $\mathcal{K}_{\text{loc}}^X$. This can be directly justified by following the argument in [22, Proposition 2.1], where it has been proven for the subordinate Brownian motions on the Sierpiński gasket. One can check that when the profile W is continuous, or when it is a non-increasing function of the Euclidean distance, then the condition (2.7) is satisfied under the assumption $W \in L^1(\mathbb{R}^d)$. Starting from Section 4 we will be interested in the Poissonian potentials with finite-range (compactly supported) profiles W , for which (2.7) holds automatically. By the range of a profile W we mean $a := \inf \{r > 0 : W(x) = 0 \text{ for Lebesgue-almost all } x \in B(0, r)^c\}$.

2.3. Random semigroups and the integrated density of states

Suppose that $W : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a profile function belonging to $\mathcal{K}_{\text{loc}}^X$ for which (2.7) holds. As indicated above, V^ω given by (2.8) belongs to $\mathcal{K}_{\text{loc}}^X$, \mathbb{Q} –almost surely. Therefore we can legitimately define the random Feynman-Kac semigroups $\{P_t^{V^\omega} : t \geq 0\}$ and $\{P_t^{U, V^\omega} : t \geq 0\}$ related to the ‘free’ process and the process killed on exiting an open, bounded and nonempty set $U \subset \mathbb{R}^d$. They consist of operators

$$P_t^{V^\omega} f(x) = \mathbf{E}_x \left[f(X_t) e^{-\int_0^t V^\omega(X_s) \, ds} \right], \quad f \in L^2(\mathbb{R}^d), \, t > 0,$$

and

$$P_t^{U, V^\omega} f(x) = \mathbf{E}_x \left[f(X_t) e^{-\int_0^t V^\omega(X_s) \, ds} \mathbf{1}_{\{\tau_U > t\}} \right], \quad f \in L^2(U), \, t > 0,$$

and admit the measurable, strictly positive, bounded and symmetric kernels $p^{V^\omega}(t, x, y)$ and $p^{U, V^\omega}(t, x, y)$, respectively. It is known that \mathbb{Q} –a.s. the semigroup $\{P_t^{V^\omega} : t \geq 0\}$ coincides with the semigroup generated by the operator $-H^\omega$, where $H^\omega = -L + V^\omega$ is the *random nonlocal Schrödinger operator* based on

the generator L of the process X , with Poissonian potential V^ω [9, 5]. The semigroup $\{P_t^{U,V^\omega} : t \geq 0\}$ corresponds then to the random nonlocal Schrödinger operator H_U^ω with exterior Dirichlet conditions on U . The operators P_t^{U,V^ω} are Hilbert-Schmidt, so that the spectrum of the operator H_U^ω is \mathbb{Q} -a.s. discrete:

$$\lambda_1(U, V^\omega) < \lambda_2(U, V^\omega) \leq \lambda_3(U, V^\omega) \dots \rightarrow \infty.$$

Again, we will single out the case of α -stable processes and denote the respective eigenvalues by $\lambda_i^{(\alpha)}(U, V^\omega)$. Similarly, P_t^V and $P_t^{U,V}$ will denote operators relative to nonrandom potentials $0 \leq V \in \mathcal{K}_{\text{loc}}^X$.

Consider now the process killed on exiting the boxes $U = U_R = (-R, R)^d$, and the random empirical measures on \mathbb{R}_+ , based on the spectra the generators of such processes, normalized by the volume:

$$(2.9) \quad \ell_R^\omega := \frac{1}{(2R)^d} \sum_{n=1}^{\infty} \delta_{\lambda_n(U_R, V^\omega)}.$$

From the maximal ergodic theorem it follows that \mathbb{Q} -a.s. the measures ℓ_R^ω are vaguely convergent as $R \rightarrow \infty$ to a nonrandom measure ℓ on \mathbb{R}_+ , called the *integrated density of states* (see e.g. [26, p. 635]). The cumulative distribution function of the measure ℓ will be denoted by $N^D(\lambda)$. The superscript D indicates that we are dealing with the Dirichlet exterior conditions (as opposed to the Neumann conditions, which are not pursued in this paper).

2.4. Notation

We say that the function g is asymptotically equivalent to the function f at infinity, which is denoted by $g \approx f$, when $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$. Likewise, when we say $f \asymp g$, we mean that there exists a constant $C \in [1, \infty)$, such that $\frac{1}{C}g(x) \leq f(x) \leq Cg(x)$ for all relevant arguments x (the range will be clear from the context). For an open set $U \subset \mathbb{R}^d$, $C_c^\infty(U)$ stands for C^∞ -functions with compact support inside U . $B(x, R)$ denotes the open Euclidean ball with center x and radius $R > 0$. We also say that a measurable function $W : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is not identically zero, if $|\{x \in \mathbb{R}^d : W(x) > 0\}| > 0$ (by $|U|$ we denote the Lebesgue measure of the set U). Important constants are denoted with upper case letters C, K, Q , possibly with subscripts. Technical constants are numbered within each proof separately as c_1, c_2, \dots .

3. The upper bound

3.1. Preliminary estimates

We start with two preliminary results. First, a lemma, proven for nonrandom potentials. Recall that the constant t_0 comes from the assumption (2.1).

Lemma 3.1. *Let $(X_t)_{t \geq 0}$ be a symmetric strong Feller Lévy process with Lévy-Khintchine exponent ψ as in (1.2) and (2.1), and let $0 \leq V \in \mathcal{K}_{\text{loc}}^X$. Then there exists a constant $C_1 = C_1(X, d)$ such that for any open, nonempty set $U \subset \mathbb{R}^d$ one has*

$$(3.1) \quad P_t^V \mathbf{1}(x) \leq C_1 |U|^{1/2} e^{-\lambda_1(U, V)(t-t_0/2)} + \mathbf{P}_x[\tau_U \leq t], \quad x \in U, \quad t > t_0/2.$$

Proof. The proof goes along standard arguments. Let U, x , and t be as in the assumptions. We have

$$\begin{aligned} P_t^V \mathbf{1}(x) &\leq \mathbf{E}_x \left[e^{-\int_0^t V(X_s) ds}; \tau_U > t \right] + \mathbf{P}_x[\tau_U \leq t] \\ &= P_t^{U, V} \mathbf{1}_U(x) + \mathbf{P}_x[\tau_U \leq t] \end{aligned}$$

and for any $R > 0$

$$\begin{aligned} P_t^{U,V} \mathbf{1}_U(x) &\leq P_t^{U,V} \mathbf{1}_{B(x,R)}(x) + P_t^{U,V} \mathbf{1}_{B(x,R)^c}(x) \\ &\leq P_{t_0/2}^{U,V} P_{t-t_0/2}^{U,V} \mathbf{1}_{B(x,R)}(x) + \mathbf{P}_x[X_t \notin B(x,R)]. \end{aligned}$$

Further,

$$\begin{aligned} P_{t_0/2}^{U,V} P_{t-t_0/2}^{U,V} \mathbf{1}_{B(x,R)} &= \left\langle p^{U,V}(t_0/2, x, \cdot), P_{t-t_0/2}^{U,V} \mathbf{1}_{B(x,R)}(\cdot) \right\rangle_{L^2(U)} \\ &\leq \|p^{U,V}(t_0/2, x, \cdot)\|_{L^2(U)} \|P_{t-t_0/2}^{U,V} \mathbf{1}_{B(x,R)}\|_{L^2(U)} \\ &\leq (p^{U,V}(t_0, x, x))^{1/2} e^{-\lambda_1(U,V)(t-t_0/2)} \|\mathbf{1}_{B(x,R)}\|_{L^2(U)} \\ &\leq c(p(t_0, 0))^{1/2} e^{-\lambda_1(U,V)(t-t_0/2)} \sqrt{|U \cap B(x, R)|}. \end{aligned}$$

Collecting these estimates we obtain

$$P_t^V \mathbf{1}(x) \leq C_1 \sqrt{|U \cap B(x, R)|} e^{-\lambda_1(U,V)(t-t_0/2)} + \mathbf{P}_x[\tau_U \leq t] + \mathbf{P}_0[|X_t| \geq R], \quad x \in U, \quad t > t_0/2.$$

To get (3.1), it is enough to take the limit $R \rightarrow \infty$. \square

In the random setting, we will need the following lemma on the mean number of eigenvalues not exceeding a given level $\lambda > 0$.

Lemma 3.2. *Let X be a symmetric strong Feller Lévy process with characteristic exponent ψ satisfying (1.2) and (2.1), and let V^ω be a Poissonian potential defined in (2.8). For $n \in \mathbb{Z}_+$ let $D_n = (-2^n, 2^n)^d$. Then for every $\lambda > 0$ we have*

$$(3.2) \quad 2^d \mathbf{E}_{\mathbb{Q}}[\#\{k \in \mathbb{Z}_+ : \lambda_k(D_n, V^\omega) \leq \lambda\}] \leq \mathbf{E}_{\mathbb{Q}}[\#\{k \in \mathbb{Z}_+ : \lambda_k(D_{n+1}, V^\omega) \leq \lambda\}], \quad n \in \mathbb{Z}_+.$$

Consequently, for any box D_n as above and any $\lambda > 0$ one has

$$(3.3) \quad \frac{1}{|D_n|} \mathbf{E}_{\mathbb{Q}}[\#\{k \in \mathbb{Z}_+ : \lambda_k(D_n, V^\omega) \leq \lambda\}] \leq N^D(\lambda).$$

Proof. Let $n \in \mathbb{Z}_+$. Denote by $\{D_n^i\}_{i=1}^{2^d}$ the collection of 2^d disjoint open boxes of the form $\mathbf{x} + (0, 2^{n+1})^d$, $\mathbf{x} \in 2^{n+1}\mathbb{Z}^d$ such that $\bigcup_{i=1}^{2^d} D_n^i \subset D_{n+1}$ and $|\bigcup_{i=1}^{2^d} D_n^i| = |D_{n+1}| = 2^{(n+2)d}$.

By using standard min-max formulas for eigenvalues (see, e.g., [35, Section 12.1]), one can check that

$$(3.4) \quad \sum_{i=1}^{2^d} \#\{k \in \mathbb{Z}_+ : \lambda_k(D_n^i, V^\omega) \leq \lambda\} \leq \#\{k \in \mathbb{Z}_+ : \lambda_k(D_{n+1}, V^\omega) \leq \lambda\}.$$

Moreover, the space homogeneity of the process together with the stationarity of the potential V^ω give

$$\mathbf{E}_{\mathbb{Q}}[\#\{k \in \mathbb{Z}_+ : \lambda_k(D_n^i, V^\omega) \leq \lambda\}] = \mathbf{E}_{\mathbb{Q}}[\#\{k \in \mathbb{Z}_+ : \lambda_k(D_n, V^\omega) \leq \lambda\}], \quad i = 1, \dots, 2^d.$$

Taking the expected value $\mathbf{E}_{\mathbb{Q}}$ on both sides of (3.4), we immediately get (3.2). \square

3.2. A general upper bound

We first introduce an auxiliary function through which we determine the typical asymptotic profile for the quenched asymptotics of the function $u^\omega(t, x)$.

For every $\alpha \in (0, 2]$, $\kappa > 0$, and a nonincreasing function $F : [1, \infty) \rightarrow [0, \infty)$ with $\lim_{r \rightarrow \infty} F(r) = 0$ we define the function $f_{F, \alpha, \kappa} : [1, \infty) \rightarrow [0, \infty)$ by

$$(3.5) \quad f_{F, \alpha, \kappa}(r) = \left((r \wedge |\log(1 \wedge F(r))|) + \frac{d}{2} \log r \right) \left(\frac{d \log r}{\kappa} \right)^{\frac{\alpha}{d}}.$$

One can directly see that for any fixed $\alpha \in (0, 2]$, $\kappa > 0$, and a given function $F(r)$ we have

$$f_{F,\alpha,\kappa}(1) = 0, \quad \lim_{r \rightarrow \infty} f_{F,\alpha,\kappa}(r) = \infty, \quad \text{moreover } f_{F,\alpha,\kappa}(r) \text{ is strictly increasing in } r.$$

In particular, the inverse function

$$(3.6) \quad h_{F,\alpha,\kappa} := [f_{F,\alpha,\kappa}]^{-1} : [0, \infty) \rightarrow [1, \infty)$$

is well defined. It is strictly increasing and satisfies

$$\lim_{t \rightarrow \infty} h_{F,\alpha,\kappa}(t) = \infty.$$

When the parameters α and κ will be fixed, they will be dropped. Observe that the function $h_{F,\alpha,\kappa}(t)$ satisfies:

$$(3.7) \quad t \left(\frac{\kappa}{d \log h_{F,\alpha,\kappa}(t)} \right)^{\frac{\alpha}{d}} = \left(h_{F,\alpha,\kappa}(t) \wedge |\log(1 \wedge F(h_{F,\alpha,\kappa}(t)))| \right) + \frac{d}{2} \log h_{F,\alpha,\kappa}(t).$$

The function $h_{F,\alpha,\kappa}(t)$ will play a central role in determining the rate of decay of the functionals considered.

In what follows we will work under the following regularity assumption **(U)** on the process X . In the condition below, the constant t_0 comes from the assumption (2.1).

(U) There are constants $C_2 > 0$, $C_3 \geq 2$, $\gamma > 0$, $r_0 \geq 1$, $t_1 \geq t_0 \vee 1$ and a nonincreasing function $F : [1, \infty) \rightarrow [0, \infty)$ such that $\lim_{r \rightarrow \infty} F(r) = 0$, for which

$$\mathbf{P}_0(|X_t| \geq r) \leq C_2 t^\gamma (F(r) \vee e^{-r}), \quad r \geq r_0 \vee C_3 t, \quad t \geq t_1.$$

The next theorem is our main result in this section.

Theorem 3.1. *Let X be a symmetric strong Feller Lévy process with characteristic exponent ψ satisfying (1.2) and (2.1) such that the assumption **(U)** holds. Let V^ω be a Poissonian potential defined in (2.8). If there exist $\alpha \in (0, 2]$ and $\kappa_0 > 0$ such that*

$$(3.8) \quad \limsup_{\lambda \rightarrow 0^+} \lambda^{d/\alpha} \log N^D(\lambda) \leq -\kappa_0,$$

then for every fixed $x \in \mathbb{R}^d$ one has

$$(3.9) \quad \limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x) - \frac{d}{2} \log h_{F,\alpha,\kappa_0}(t)}{g(t)} \leq -\left(\frac{\kappa_0}{d}\right)^{\alpha/d}, \quad \mathbb{Q} - a.s.,$$

where the function $h_{F,\alpha,\kappa}$ is defined in (3.6) with F given by **(U)** and $g(t) := t/(\log h_{F,\alpha,\kappa}(t))^{\alpha/d}$.

Proof. Fix $x \in \mathbb{R}^d$ and let r_0 , t_1 , γ , α , κ_0 and F be as in the assumptions. Specifically, we may and do assume that $r_0 \geq 1$ is so large that $F(r) \leq 1$ for $r \geq r_0$. We will write h for h_{F,α,κ_0} . By Lemma 3.1, for every $t \geq t_0/2$ and every open set $U \ni x$, we have

$$(3.10) \quad u^\omega(t, x) = P_t^{V^\omega} \mathbf{1}(x) \leq C_1 \sqrt{|U|} e^{-\lambda_1(U, V^\omega)(t-t_0/2)} + \mathbf{P}_x[\tau_U \leq t].$$

In particular, we can choose $U = U_{2R} = (-2R, 2R)^d$, where R is so large that

$$(3.11) \quad R > |x| \vee r_0 \vee C_3 t \quad \text{and} \quad t \geq t_1.$$

Now, since for this choice of U we have $B(x, R) \subset U$, from the Lévy inequality and assumption **(U)** we obtain:

$$(3.12) \quad \begin{aligned} \mathbf{P}_x[\tau_U \leq t] &\leq \mathbf{P}_x[\tau_{B(x,R)} \leq t] = \mathbf{P}_0[\tau_{B(0,R)} \leq t] = \mathbf{P}_0[\sup_{s \in [0,t]} |X_s| \geq R] \\ &\leq 2\mathbf{P}_0[|X_t| \geq R] \leq 2C_2 t^\gamma e^{-(R \wedge |\log(F(R)))|}. \end{aligned}$$

We now estimate $\lambda_1(U_{2R}, V^\omega)$ for large R . Inequality (3.3) from Lemma 3.2 holds for dyadic boxes D_n and reads:

$$\frac{1}{2^{(n+1)d}} \mathbf{E}_{\mathbb{Q}} [\# \{k \geq 1 : \lambda_k(D_n, V^\omega) \leq \lambda\}] \leq N^D(\lambda), \quad \lambda > 0, \quad n \in \mathbb{Z}_+.$$

Running the argument from [11, (2.3)-(2.6)] with $\phi(r) = \kappa_0 r^{d/\alpha}$ and the sequence $t_n = 2^n$, from the assumption (3.8), we get that for every $\varepsilon \in (0, 1)$, \mathbb{Q} -almost surely we can find $R_\varepsilon = R_\varepsilon(\omega) > 1$ such that for every $R \geq R_\varepsilon$

$$(3.13) \quad \lambda_1(U_{2R}, V^\omega) \geq (1 - \varepsilon) \left(\frac{\kappa_0}{d \log R} \right)^{\alpha/d}.$$

Piecing together (3.10), (3.12), and (3.13) we get that for every $\varepsilon \in (0, 1)$, \mathbb{Q} -a.s. there exists $R_\varepsilon > 1$ such that for all t and R satisfying $R \geq R_\varepsilon$ and (3.11) one has

$$u^\omega(t, x) \leq C_1(4R)^{d/2} e^{-(1-\varepsilon)(t-t_0/2)} \left(\frac{\kappa_0}{d \log R} \right)^{\alpha/d} + 2C_2 t^\gamma e^{-(R \wedge |\log(F(R))|)},$$

and further,

$$(3.14) \quad \log u^\omega(t, x) \leq -\min \left\{ (1 - \varepsilon) \left(t - \frac{t_0}{2} \right) \left(\frac{\kappa_0}{d \log R} \right)^{\alpha/d} - \frac{d}{2} \log R, R \wedge |\log(F(R))| \right\} + \gamma \log t + c_1,$$

with an absolute constant $c_1 \geq 0$.

Let now $h(t)$ be given by (3.6). As $h(t) \rightarrow \infty$ when $t \rightarrow \infty$, \mathbb{Q} -a.s. there exists $t_2 \geq t_1$ large enough so that for every $t \geq t_2$ the condition (3.11) holds with $R = h(t) \vee C_3 t$, and moreover $R \geq R_\varepsilon(\omega)$. Thus we may substitute in (3.14) the value

$$(3.15) \quad R = R(t) := h(t) \vee C_3 t.$$

Next, from the definition of $h(t)$, (3.7), and the monotonicity of f_{F, α, κ_0} , we see that

$$(3.16) \quad (h(t) \vee C_3 t) \wedge \left| \log \left(1 \wedge F(h(t) \vee C_3 t) \right) \right| \geq t \left(\frac{\kappa_0}{d \log (h(t) \vee C_3 t)} \right)^{\frac{\alpha}{d}} - \frac{d}{2} \log (h(t) \vee C_3 t)$$

with equality when $h(t) \geq C_3 t$. We finally obtain that for all $t > t_2$ we have

$$\begin{aligned} \log u^\omega(t, x) &\leq - \left((1 - \varepsilon)(t - t_0/2) \left(\frac{\kappa_0}{d \log (h(t) \vee C_3 t)} \right)^{\alpha/d} - \frac{d}{2} \log (h(t) \vee C_3 t) \right) + \gamma \log t + c_1 \\ &\leq - \left((1 - \varepsilon)(t - t_0/2) \left(\frac{\kappa_0}{d \log (h(t) \vee C_3 t)} \right)^{\alpha/d} - \frac{d}{2} \log h(t) \right) + \left(\gamma + \frac{d}{2} \right) \log t + c_2, \end{aligned}$$

with absolute constants $c_1, c_2 \geq 0$, for \mathbb{Q} -almost all ω .

To complete the proof, it remains to show that

$$(3.17) \quad \log t = o(g(t)) \quad \text{as } t \rightarrow \infty,$$

where $g(t) := t/(\log h(t))^{\alpha/d}$. This is obvious when $h(t) \leq C_3 t$, and when $h(t) \geq C_3 t$, then from (3.7) we have

$$t \left(\frac{\kappa_0}{d \log h(t)} \right)^{\alpha/d} - \frac{d}{2} \log h(t) > 0, \quad \text{so that } h(t) \leq e^{c_3 t^{\frac{d}{d+\alpha}}},$$

for some $c_3 > 0$ and (3.17) follows.

We conclude that \mathbb{Q} -almost surely

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x) - \frac{d}{2} \log h(t)}{g(t)} \leq -(1 - \varepsilon) \left(\frac{\kappa_0}{d} \right)^{\alpha/d}, \quad \text{with } g(t) = \frac{t}{(\log h(t))^{\alpha/d}}.$$

Letting $\varepsilon \rightarrow 0$ through rational numbers, we get (3.9). The proof is complete. \square

The next corollary will eventually enable us to obtain, for certain processes, the existence of $\lim_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)}$.

Corollary 3.1. *Let the assumptions of Theorem 3.1 above be satisfied. More specific, let **(U)** hold with a function F . If there exists $Q_1 \in (0, \infty]$ such that*

$$\liminf_{r \rightarrow \infty} \frac{|\log F(r)|}{\log r} \geq Q_1$$

then

$$(3.18) \quad \limsup_{t \rightarrow \infty} \frac{\log h_{F, \alpha, \kappa_0}(t)}{g(t)} \leq \frac{2}{2Q_1 + d} \left(\frac{\kappa_0}{d} \right)^{\alpha/d}$$

and, consequently, for every fixed $x \in \mathbb{R}^d$, one has

$$(3.19) \quad \limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \leq - \left(\frac{\kappa_0}{d} \right)^{\alpha/d} \left(1 - \frac{d}{d + 2Q_1} \right), \quad \mathbb{Q} - a.s.$$

In particular, when $\lim_{r \rightarrow \infty} \frac{|\log F(r)|}{\log r} = \infty$, then

$$(3.20) \quad \limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \leq - \left(\frac{\kappa_0}{d} \right)^{\alpha/d}, \quad \mathbb{Q} - a.s.$$

Proof. The assumptions give that for any $0 < \tilde{Q}_1 < Q_1$ there exists r_0 such that for $r > r_0$

$$f_{F, \alpha, \kappa_0}(r) \geq \left(r \wedge (\tilde{Q}_1 \log r) + \frac{d}{2} \log r \right) \left(\frac{d}{\kappa_0} \log r \right)^{\alpha/d},$$

which is equivalent to saying that for sufficiently large r

$$f_{F, \alpha, \kappa_0}(r) \geq \left(\tilde{Q}_1 + \frac{d}{2} \right) \left(\frac{d}{\kappa_0} \right)^{\alpha/d} (\log r)^{1 + \alpha/d}$$

or, for sufficiently large t ,

$$h_{F, \alpha, \kappa_0}(t) \leq e^{\left(\frac{2}{2\tilde{Q}_1 + d} \left(\frac{\kappa_0}{d} \right)^{\alpha/d} t \right)^{\frac{d}{d + \alpha}}},$$

and further

$$0 \leq \frac{\log h_{F, \alpha, \kappa_0}(t)}{g(t)} \leq \frac{(\log h_{F, \alpha, \kappa_0}(t))^{\frac{\alpha + d}{d}}}{t} \leq \frac{2}{2\tilde{Q}_1 + d} \left(\frac{\kappa_0}{d} \right)^{\alpha/d}.$$

This means that

$$0 \leq \limsup_{t \rightarrow \infty} \frac{\log h_{F, \alpha, \kappa_0}(t)}{g(t)} \leq \frac{2}{2\tilde{Q}_1 + d} \left(\frac{\kappa_0}{d} \right)^{\alpha/d}.$$

Letting $\tilde{Q}_1 \nearrow Q_1$ we get (3.18). Statements (3.19) and (3.20) follow directly from (3.9). \square

4. The lower bound for regularly distributed Lévy measures

As indicated in the Introduction, the argument deriving the quenched asymptotic lower bound directly from the lower asymptotics of the IDS seems to be not obvious in the non-diffusion case. Instead, we estimate $u^\omega(t, x)$ directly. In this part (similarly as in [32, 11]), we require the potential profile W to be bounded and compactly supported. As usual in problems of this kind, we first prove that \mathbb{Q} -almost surely there exist sufficiently large regions without potential interaction, then we force the process to go to this region and then stay there for a long enough time. This behaviour will be described analytically.

4.1. Typical potential configuration

Let $\varepsilon > 0$ be given. For a given number $r > 0$, let $M^\varepsilon(r)$ be defined by

$$(4.1) \quad M^\varepsilon(r) := \left(\frac{d}{\omega_d \rho(1 + \varepsilon)} \right)^{\frac{2}{d} + 2} r^{-2d-2} e^{\frac{\omega_d \rho(1 + \varepsilon)}{d} r^d},$$

where ω_d denotes the volume of the unit ball in \mathbb{R}^d . We have a lemma.

Lemma 4.1. *Let $\varepsilon > 0$, $a > 0$, and let r and $M^\varepsilon(r)$ be related by (4.1). Then \mathbb{Q} -almost surely, there exists $r_0 > 0$ such that for $r > r_0$ the set $(-M^\varepsilon(r), M^\varepsilon(r))^d \setminus (-r, r)^d$ contains a ball with radius r whose a -neighbourhood is free of Poisson points.*

Proof. Assume first that $r = m \in \mathbb{Z}$. Then

$$(4.2) \quad \mathbb{Q}[\text{the ball } B(0, m + a) \text{ contains no Poisson points}] = e^{-\omega_d \rho(m+a)}.$$

The equality (4.2) is true for any translate of $B(0, m + a)$. Inside the box $(-M^{\varepsilon/2}(m), M^{\varepsilon/2}(m))^d$, one can pack $a_m := \left\lceil \frac{2M^{\varepsilon/2}(m)}{2(m+a)} \right\rceil^d$ disjoint boxes of size $2m$ that are $(2a)$ -separated. Open balls of radius $(m + a)$, ‘concentric’ with those boxes are disjoint. As the realizations of the cloud over disjoint sets are independent random variables, the probability that each such (small) ball contains at least one Poisson point (denote this event by \mathcal{A}_m^0) equals to

$$\mathbb{Q}[\mathcal{A}_m^0] = (1 - e^{-\omega_d \rho(m+a)^d})^{a_m}.$$

We would like to produce a ball with radius $(m + a)$ that is both: free of Poisson points and separated from zero, so that we exclude from our considerations the boxes whose closure might contain zero, at most 2^d of them. Let \mathcal{A}_m be the event that ‘every small ball from $(-M^{\varepsilon/2}(m), M^{\varepsilon/2}(m))^d \setminus (-m - a, m + a)^d$, arising as above, contains a Poisson point’, then

$$\mathbb{Q}[\mathcal{A}_m] = \left(1 - e^{-\omega_d \rho(m+a)^d} \right)^{a_m - 2^d}.$$

Using an elementary inequality $(1 - x) \leq e^{-x}$ we can write

$$\mathbb{Q}[\mathcal{A}_m] \leq \exp \left\{ -e^{-\omega_d \rho(m+a)^d} \left(\frac{M^{\varepsilon/2}(m)}{(m+a)} \right)^d (1 - o(1)) \right\}, \quad \text{for } m \rightarrow \infty,$$

and the expression in the exponent is equal to (recall $r = m$)

$$-(1 - o(1)) e^{\omega_d \rho \frac{\varepsilon}{2} m^d (1 - o(1))}, \quad m \rightarrow \infty$$

so that

$$\sum_{m=1}^{\infty} \mathbb{Q}[\mathcal{A}_m] < \infty.$$

From the Borel-Cantelli lemma we get that \mathbb{Q} -almost surely there is a number $m_0 > 0$ such that for $m > m_0$ the set $(-M^{\varepsilon/2}(m), M^{\varepsilon/2}(m))^d \setminus (-m-a, m+a)^d$ contains a ball of radius $(m+a)$ free of Poisson points. In particular, if $r > m_0$ is a real number, then we can find an empty ball of size $[r] + 1 + a$ included in the big box $(-M^{\varepsilon/2}([r] + 1), M^{\varepsilon/2}([r] + 1))^d$ and separated from zero. Since

$$M^{\varepsilon/2}([r] + 1) \leq M^{\varepsilon}(r) \quad \text{for sufficiently large } r,$$

the lemma follows. \square

We also quote Lemma 3.2 from [11] (we have $\alpha = 1$ in present case).

Lemma 4.2. *Suppose that the profile function W is compactly supported and bounded. Then \mathbb{Q} -almost-surely, for sufficiently large R one has*

$$(4.3) \quad \sup_{x \in (-R, R)^d} V_{\omega}(x) \leq 3d \log R.$$

4.2. A general lower bound

Let $R > R_0 > 0$ be given and let $p^{U_R}(t, x, y)$ be the Dirichlet kernel of our process $(X_t)_{t \geq 0}$ in the box $U_R := (-R, R)^d$. To begin with, we introduce the following notation:

$$(4.4) \quad G(R_0, R) := \inf_{\frac{R_0}{2} \leq |y| \leq \frac{R}{2}} p^{U_R}(1, 0, y).$$

Also, recall that $\lambda_1^{(\alpha)}(B(0, 1))$ is the principal Dirichlet eigenvalue of the symmetric α -stable process defined by (1.5) in the unit ball $B(0, 1)$ and ω_d is the volume of this ball.

We now present our main theorem in this section.

Theorem 4.1. *Let X be a symmetric strong Feller Lévy process with characteristic exponent ψ satisfying (1.2) and (2.1). Moreover, suppose that there exist $\alpha \in (0, 2]$ and $K > 0$ such that*

$$(4.5) \quad \lambda_1^{\psi}(B(0, R)) \leq KR^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + o(R^{-\alpha}), \quad R \rightarrow \infty,$$

and that V^{ω} is a Poissonian potential defined in (2.8) with bounded profile W of finite range. Then for any $x \in \mathbb{R}^d$, $\kappa, R_0 > 0$ and any nonincreasing function $F : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{r \rightarrow \infty} F(r) = 0$, \mathbb{Q} -almost surely,

$$\liminf_{t \rightarrow \infty} \frac{\log u^{\omega}(t, x) + 3d \log h_{F, \alpha, \kappa}(t) + |\log j_{R_0, F, \alpha, \kappa}(t)|}{g(t)} \geq -K \left(\frac{\omega_d \rho}{d} \right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1)),$$

where $h_{F, \alpha, \kappa}(t)$ was defined by (3.6), $g(t) = \frac{t}{(\log h_{F, \alpha, \kappa}(t))^{\alpha/d}}$, and $j_{R_0, F, \alpha, \kappa}(t) := G \left(R_0, \frac{2\sqrt{d} h_{F, \alpha, \kappa}(t)}{(\log h_{F, \alpha, \kappa}(t))^{\frac{2}{d}+2}} \right)$ (the function $j_{R_0, F, \alpha, \kappa}(t)$ is well-defined for large t 's).

Proof. For simplicity, we run the proof for $x = 0$ only; for a general $x \in \mathbb{R}^d$ the proof is identical. Let $\kappa > 0$ and $R_0 > 0$ be given. As in the proof of Theorem 3.1, we will write h for $h_{F, \alpha, \kappa}$. Let $\varepsilon > 0$ be given and let a be the range of the potential profile W , then for $t > 0$ let $m(t)$ and $M(t)(= M^{\varepsilon}(m(t)))$ be related by (4.1). The potential range a is fixed so it does not enter the notation. For the time being we require only that $m(t) \rightarrow \infty$ when $t \rightarrow \infty$. Eventually, the number $m(t)$ will be chosen of order $h(t)$ from (3.6), but in such a manner that $M^{\varepsilon}(m)$ will bear no ε -dependence.

Pick ω outside the exceptional sets from Lemmas 4.1 and 4.2. Let B_t be the open ball of radius $m(t)$ whose a -neighbourhood contains no Poisson points, obtained from the statement of Lemma 4.1. As there is no interaction with the potential inside this ball, we have that $p^{B_t, V^{\omega}}(\cdot, \cdot, \cdot) = p^{B_t}(\cdot, \cdot, \cdot)$, and consequently

$\lambda_1^\psi(B_t, V^\omega) = \lambda_1^\psi(B_t)$ (recall that $\lambda^\psi(U, V^\omega)$ and $\lambda_1^\psi(U)$ denote the principal Dirichlet eigenvalue of the process in U under the influence of the potential V^ω , or without potential interaction, respectively).

Let ϕ be the normalized, positive Dirichlet L^2 -eigenfunction, supported in B_t , corresponding to this principal eigenvalue. For sufficiently large t we have the following chain of inequalities:

$$\begin{aligned}
u^\omega(t, 0) &= \int_{\mathbb{R}^d} p^{V^\omega}(t, 0, x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{V^\omega}(1, 0, y) p^{V^\omega}(t-1, y, x) dy dx \\
&\geq \int_{B_t} \int_{B_t} p^{V^\omega}(1, 0, y) p^{V^\omega}(t-1, y, x) dy dx \\
&\geq \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right) \int_{B_t} \left(\int_{B_t} p^{V^\omega}(t-1, y, x) \frac{\phi(x)}{\|\phi\|_\infty} dx \right) dy \\
&\geq \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right) \int_{B_t} \left(\int_{B_t} p^{V^\omega, B_t}(t-1, y, x) \frac{\phi(x)}{\|\phi\|_\infty} dx \right) dy \\
&= \frac{1}{\|\phi\|_\infty} \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right) \int_{B_t} e^{-\lambda_1^\psi(V^\omega, B_t)(t-1)} \phi(y) dy \\
&\geq \frac{1}{\|\phi\|_\infty^2} \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right) e^{-\lambda_1^\psi(B_t)(t-1)} \int_{B_t} \phi(y)^2 dy \\
&\geq \frac{1}{\|\phi\|_\infty^2} e^{-t\lambda_1^\psi(B_t)} \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right).
\end{aligned}$$

From the translation invariance of the process and assumption (4.5) we see that

$$(4.6) \quad \lambda_1^\psi(B_t) \leq K m(t)^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + o(m(t)^{-\alpha}), \quad \text{as } t \rightarrow \infty.$$

Also, it is classical to see that $\|\phi\|_\infty \leq c_1 \lambda_1^\psi(B_t)^{d/(2\alpha)}$, which from (4.6) can be estimated as $c_2 m(t)^{-d/2}$, so that $\|\phi\|_\infty^2 m(t)^d \leq c_3$. The chain of inequalities continues as

$$(4.7) \quad \geq c_4 \left(\inf_{y \in B_t} p^{V^\omega}(1, 0, y) \right) e^{-t\lambda_1^\psi(B_t)} m(t)^d.$$

To estimate the infimum of the kernel $p^{V^\omega}(1, 0, y)$ for $y \in B_t$, we take $J_t = (-2\sqrt{d}M(t), 2\sqrt{d}M(t))^d$. For $y \in B_t$ one has $y \in (-M(t), M(t))^d \setminus (-m(t), m(t))^d$ so that for sufficiently large t one has $R_0 \leq y \leq \sqrt{d}M(t)$. Using (4.3) and (4.4) we can write:

$$\begin{aligned}
(4.8) \quad p^{V^\omega}(1, 0, y) &\geq p^{J_t, V^\omega}(1, 0, y) \geq e^{-3d \log(2\sqrt{d}M(t))} p^{J_t}(1, 0, y) \\
&\geq c_5 e^{-3d \log(2\sqrt{d}M(t))} G(R_0, 2\sqrt{d}M(t)).
\end{aligned}$$

Inserting these estimates inside (4.7) and using (4.6) again, we obtain that \mathbb{Q} -a.s., for sufficiently large t :

$$\begin{aligned}
u^\omega(t, 0) &\geq c_6 \exp \left\{ -3d \log(2\sqrt{d}M(t)) - |\log G(R_0, 2\sqrt{d}M(t))| - K t m(t)^{-\alpha} (\lambda_1^{(\alpha)}(B(0, 1)) + o(1)) + d \log m(t) \right\}.
\end{aligned}$$

At this point we declare the scale $m(t)$. Recall that all this reasoning is performed for a fixed number $\varepsilon > 0$. Set $m(t) = m_\varepsilon(t)$ to be the solution of the equation (unique for large t)

$$(4.9) \quad \omega_d \rho(1 + \varepsilon) (m_\varepsilon(t))^d = d \log h(t),$$

where $h(t)$ was given by (3.6). Consequently, using (4.1),

$$M(t) = M^\varepsilon(m_\varepsilon(t)) = \frac{h(t)}{(\log h(t))^{\frac{2}{d}+2}}.$$

It follows

$$(4.10) \quad \log u^\omega(t, 0) + 3d \log \left(\frac{2\sqrt{d}h(t)}{(\log h(t))^{\frac{2}{d}+2}} \right) + \left| \log G \left(R_0, \frac{2\sqrt{d}h(t)}{(\log h(t))^{\frac{2}{d}+2}} \right) \right| \\ \geq -K t m_\varepsilon(t)^{-\alpha} (\lambda_1^{(\alpha)}(B(0, 1)) + o(1)) + d \log m_\varepsilon(t) + O(1), \quad t \rightarrow \infty.$$

Since $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, it is immediate that $m_\varepsilon(t) \rightarrow \infty$ when $t \rightarrow \infty$. Due to (4.9) we get

$$\lim_{t \rightarrow \infty} \left(-\frac{t(m_\varepsilon(t))^{-\alpha}}{g(t)} \right) = -\left(\frac{\omega_d \rho(1 + \varepsilon)}{d} \right)^{\alpha/d} \quad \text{and} \quad d \log m_\varepsilon(t) = \log \log h(t) + O(1) \quad \text{when } t \rightarrow \infty.$$

Further, from the relation (3.7) defining h , we see that

$$g(t) = \frac{t}{(\log h(t))^{\alpha/d}} \geq c_7 \log h(t).$$

Consequently, for sufficiently large t we get

$$0 \leq \frac{\log \log h(t)}{g(t)} \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

These properties give that, \mathbb{Q} -almost surely,

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, 0) + 3d \log h(t) + |\log j_{R_0, F, \alpha, \kappa}(t)|}{g(t)} \geq -K \left(\frac{\omega_d \rho(1 + \varepsilon)}{d} \right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1)).$$

Letting $\varepsilon \rightarrow 0$ through rationals gives the statement. \square

The next corollary gives a direct lower bound for $\liminf \frac{\log u^\omega(t, x)}{g(t)}$, similar to that in Corollary 3.1.

Corollary 4.1. *Let the assumptions of the above theorem be satisfied. In particular, let F and G be the monotone functions appearing in its statement. If there exist $Q_1 \in (0, \infty]$ and $Q_2 \in [0, \infty)$ such that*

$$(4.11) \quad \liminf_{r \rightarrow \infty} \frac{|\log F(r)|}{\log(r)} \geq Q_1$$

and

$$(4.12) \quad \limsup_{r \rightarrow \infty} \frac{\left| \log G \left(R_0, \frac{2\sqrt{d}r}{(\log r)^{\frac{2}{d}+2}} \right) \right|}{(r \wedge |\log F(r)|) + (d/2) \log r} \leq Q_2,$$

then

$$(4.13) \quad \limsup_{t \rightarrow \infty} \frac{\log h_{F, \alpha, \kappa}(t)}{g(t)} \leq \frac{2}{2Q_1 + d} \left(\frac{\kappa}{d} \right)^{\alpha/d} \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{|\log j_{R_0, F, \alpha, \kappa}(t)|}{g(t)} \leq Q_2 \left(\frac{\kappa}{d} \right)^{\alpha/d}$$

and, consequently, for every fixed $x \in \mathbb{R}^d$, one has

$$(4.14) \quad \liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \geq -K \left(\frac{\omega_d \rho}{d} \right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1)) - \left(\frac{6d}{d + 2Q_1} + Q_2 \right) \left(\frac{\kappa}{d} \right)^{\alpha/d}, \quad \mathbb{Q} - a.s.$$

In particular, when the assumptions (4.11) and (4.12) hold with $Q_1 = \infty$ and $Q_2 = 0$, then

$$(4.15) \quad \liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \geq -K \left(\frac{\omega_d \rho}{d} \right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1)).$$

Proof. The first bound in (4.13) follows from (4.11) exactly by the same argument as in Corollary 3.1. To prove the second bound in (4.13) we write

$$\frac{|\log j_{R_0}(t)|}{g(t)} = \frac{\left| \log G \left(R_0, \frac{2\sqrt{d}h_{F, \alpha, \kappa}(t)}{(\log h_{F, \alpha, \kappa}(t))^{\frac{2}{d}+2}} \right) \right|}{(d/\kappa)^{\alpha/d} ((h_{F, \alpha, \kappa}(t) \wedge |\log F(h_{F, \alpha, \kappa}(t))|) + (d/2) \log h_{F, \alpha, \kappa}(t))},$$

The desired bound immediately follows from (3.7) once we recall that $h(t) \rightarrow \infty$ when $t \rightarrow \infty$. \square

5. Discussion of specific cases

We will apply the general results of previous sections to some particular processes, for which the assumptions of Theorems 3.1 and 4.1 hold true. Throughout this section we will work under the assumption that the Lévy-Khinchine exponent ψ is close to the characteristic exponent of a symmetric α -stable process near the origin. More precisely, we assume the following condition.

(C) One has

$$\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha), \quad \text{when } |\xi| \rightarrow 0,$$

where $\psi^{(\alpha)}$ is given by (1.5)-(1.6) for some $\alpha \in (0, 2]$ and satisfies $\inf_{|\xi|=1} \psi^{(\alpha)}(\xi) > 0$.

We will also assume some regularity on the behaviour of ψ at infinity, a kind of Hartman-Wintner condition:

$$(5.1) \quad \frac{\psi(\xi)}{(\log |\xi|)^2} \rightarrow \infty \quad \text{as } |\xi| \rightarrow \infty.$$

Observe that under this assumption $e^{-t\sqrt{\psi(\cdot)}} \in L^1(\mathbb{R}^d)$, for every $t > 0$. In particular, $e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^d)$, $t > 0$, and (2.1) holds for every $t_0 > 0$.

Under these assumptions, in the paper [27] the annealed asymptotics of $u^\omega(t, x)$ was proven, and also in [26] the behaviour of the integrated density of states, $N^D(\lambda)$, was established. We have the following.

Theorem 5.1. [26, Theorem 6.2] *Let X be a symmetric Lévy process whose characteristic exponent ψ satisfies (C) and (5.1), and let V^ω be a Poissonian potential defined in (2.8) with nonnegative and nonidentically zero $\mathcal{K}_{\text{loc}}^X$ -class profile W satisfying the condition $W(x) = o(|x|^{-d-\alpha})$, $|x| \rightarrow \infty$. Then*

$$(5.2) \quad \lim_{\lambda \rightarrow 0^+} \lambda^{d/\alpha} \log N^D(\lambda) = -\rho(\lambda_{(\alpha)})^{d/\alpha}.$$

The constant $\lambda_{(\alpha)}$ is given by the variational formula

$$(5.3) \quad \lambda_{(\alpha)} = \inf_G \lambda_1^{(\alpha)}(G),$$

where the infimum is taken over all open sets $G \subset \mathbb{R}^d$ of unit Lebesgue measure.

Theorem 6.2 in [26] has been proven for continuous profiles W , but its proof also applies to the local Kato-class case.

Moreover, it follows from the Faber-Krahn isoperimetric inequality (see, e.g. [10, Lemma 3.13] and [4, Theorem 3.5]) that when the process $X^{(\alpha)}$ is isotropic, then the infimum in (5.3) is attained on the ball of radius $r_d = \omega_d^{-1/d}$ (ω_d is the volume of the unit ball) and is equal to $\omega_d^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1))$.

Theorem 5.1 above states that (C) and (5.1) are sufficient conditions for the validity of (3.8), which is the main assumption of Theorem 3.1. We now show that when (C) holds, then also the quasi-scaling of principal eigenvalues needed in Theorem 4.1 holds true. The following proposition takes care of that.

Proposition 5.1. *Suppose that X is a Lévy process such that condition (C) is satisfied with certain $\alpha \in (0, 2]$. Then also (4.5) holds true, with α the same as that in (C) and any $K > 1$. More precisely, for any fixed $K > 1$ one has*

$$\lambda_1^\psi(B(0, R)) \leq KR^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + o(R^{-\alpha}), \quad R \rightarrow \infty.$$

Proof. Suppose that (C) is true; let $\psi^{(\alpha)}$ be the Lévy-Khinchine exponent of the symmetric α -stable process $X^{(\alpha)}$ appearing in this condition. Denote $\bar{\psi}(\xi) = \psi(\xi) - \psi^{(\alpha)}(\xi)$, so that for fixed $R > 0$

$$\psi\left(\frac{\xi}{R}\right) = \psi^{(\alpha)}\left(\frac{\xi}{R}\right) + \bar{\psi}\left(\frac{\xi}{R}\right).$$

For any given $u \in C_c^\infty(\mathbb{R}^d)$ let $u_R(x) = R^{-d/2}u(\frac{x}{R})$. Then

$$\begin{aligned} \mathcal{E}(u_R, u_R) &= \int_{\mathbb{R}^d} \psi(\xi) |\widehat{u_R}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} R^d \psi(\xi) |\widehat{u}(R\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} \psi\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} \psi^{(\alpha)}\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi \\ &= \mathcal{E}^{(\alpha)}(u_R, u_R) + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi \\ (5.4) \quad &= R^{-\alpha} \mathcal{E}^{(\alpha)}(u, u) + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi. \end{aligned}$$

We have assumed that $\bar{\psi}(\xi) = o(|\xi|^\alpha)$ when $|\xi| \rightarrow 0$. Therefore, since both $0 \leq \psi(\xi), \psi^{(\alpha)}(\xi) \leq c|\xi|^2$ when $|\xi| > 1$, we get that there is $c_1 > 0$ for which

$$|\bar{\psi}(\xi)| \leq c_1(|\xi|^\alpha + |\xi|^2), \quad \xi \in \mathbb{R}^d,$$

so that

$$\left| R^\alpha \bar{\psi}\left(\frac{\xi}{R}\right) \right| \leq c_1(|\xi|^\alpha + |\xi|^2), \quad \xi \in \mathbb{R}^d, R \geq 1.$$

Moreover, $R^\alpha \bar{\psi}(\xi/R) \rightarrow 0$ as $R \rightarrow \infty$, for every fixed $\xi \in \mathbb{R}^d$. Since $u \in C_c^\infty(\mathbb{R}^d)$, the integral $\int_{\mathbb{R}^d} |\xi|^2 |\widehat{u}(\xi)|^2 d\xi$ is finite, and from the dominated convergence theorem we obtain that

$$(5.5) \quad \lim_{R \rightarrow \infty} R^\alpha \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{u}(\xi)|^2 d\xi = 0.$$

Let now $\phi^{(\alpha)}$ be the first eigenfunction of the generator of the process $X^{(\alpha)}$ killed outside $B(0, 1)$. Clearly, $\phi^{(\alpha)} \in \mathcal{D}(\mathcal{E}^{(\alpha)})$ and $\lambda_1^{(\alpha)}(B(0, 1)) = \mathcal{E}^{(\alpha)}(\phi^{(\alpha)}, \phi^{(\alpha)})$. Also, let $\{\varphi_\delta\}_{\delta>0} \subset C_c^\infty(\mathbb{R}^d)$ be a family of mollifiers such that $\text{supp } \varphi_\delta \subseteq B(0, \delta)$. Denote $\phi_\delta^{(\alpha)} = (\phi^{(\alpha)} * \varphi_\delta) / \|\phi^{(\alpha)} * \varphi_\delta\|_2$. Then, for every $\delta > 0$, $\phi_\delta^{(\alpha)} \in C_c^\infty(\mathbb{R}^d)$, $\|\phi_\delta^{(\alpha)}\|_2 = 1$ and $\text{supp } \phi_\delta^{(\alpha)} \subseteq B(0, 1 + \delta)$. Moreover, $\mathcal{E}^{(\alpha)}(\phi_\delta^{(\alpha)}, \phi_\delta^{(\alpha)}) \rightarrow \mathcal{E}^{(\alpha)}(\phi^{(\alpha)}, \phi^{(\alpha)}) = \lambda_1^{(\alpha)}(B(0, 1))$ as $\delta \rightarrow 0^+$.

Suppose that $K > 1$ is fixed, and choose $\delta > 0$ small enough so that

$$\mathcal{E}^{(\alpha)}(\phi_\delta^{(\alpha)}, \phi_\delta^{(\alpha)}) \leq K(1 + \delta)^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)).$$

Then, from (5.4) it follows that for every $R > 1$

$$\begin{aligned} \lambda_1^\psi(B(0, (1 + \delta)R)) &\leq \mathcal{E}((\phi_\delta^{(\alpha)})_R, (\phi_\delta^{(\alpha)})_R) = \frac{1}{R^\alpha} \mathcal{E}^{(\alpha)}(\phi_\delta^{(\alpha)}, \phi_\delta^{(\alpha)}) + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{\phi_\delta^{(\alpha)}}(\xi)|^2 d\xi \\ &\leq \frac{K}{((1 + \delta)R)^\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{\xi}{R}\right) |\widehat{\phi_\delta^{(\alpha)}}(\xi)|^2 d\xi \end{aligned}$$

(the first inequality follows by the standard variational formula for the principal eigenvalue). Finally, by substituting $\tilde{R} = (1 + \delta)R$, we get

$$(5.6) \quad \lambda_1^\psi(B(0, \tilde{R})) \leq K \tilde{R}^{-\alpha} \lambda_1^{(\alpha)}(B(0, 1)) + \int_{\mathbb{R}^d} \bar{\psi}\left(\frac{(1 + \delta)\xi}{\tilde{R}}\right) |\widehat{\phi_\delta^{(\alpha)}}(\xi)|^2 d\xi.$$

The statement follows now from (5.5) and (5.6). \square

We now provide some reasonable and easy-to-check sufficient conditions under which the basic asymptotic assumption **(C)** holds true.

Proposition 5.2. *Let X be a Lévy process determined by the Lévy-Khintchine exponent ψ as in (1.2) with Gaussian coefficient $A = (a_{ij})_{1 \leq i, j \leq d}$ and Lévy measure ν . The following hold.*

(i) *If ν has second moment finite, i.e.*

$$\int_{\mathbb{R}^d} |z|^2 \nu(dz) < \infty,$$

then

$$\psi(\xi) = \xi \cdot \tilde{A}\xi + o(|\xi|^2) \quad \text{as } |\xi| \rightarrow 0,$$

where $\tilde{A} = (\tilde{a}_{ij})_{1 \leq i, j \leq d}$ with $\tilde{a}_{ij} = a_{ij} + \frac{1}{2} \int_{\mathbb{R}^d} y_i y_j \nu(dy)$.

(ii) *If there exist $\alpha \in (0, 2)$ and a symmetric finite measure n on the unit sphere S^{d-1} such that*

$$r^2 \int_{1 < |z| \leq 1/r} |z|^2 |\nu - \nu^{(\alpha)}|(dz) = o(r^\alpha) \quad \text{and} \quad |\nu - \nu^{(\alpha)}|(B(0, 1/r)^c) = o(r^\alpha) \quad \text{as } r \rightarrow 0,$$

with $\nu^{(\alpha)}(dr d\varphi) = n(d\varphi) r^{-1-\alpha} dr$, then

$$\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha) \quad \text{as } |\xi| \rightarrow 0,$$

where $\psi^{(\alpha)}(\xi) = \int_0^\infty \int_{S^{d-1}} (1 - \cos(\xi \cdot r\varphi)) n(d\varphi) \frac{dr}{r^{\alpha+1}}$.

Proof. Knowing that $\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) \nu(dy)$, and writing down the Taylor expansion of the function $\cos s$ at 0 we get

$$\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d} (\xi \cdot y)^2 \left[\int_0^1 (1-t) \cos(t\xi \cdot y) dt \right] \nu(dy).$$

The first assertion follows from the dominated convergence theorem together with the finiteness of the second moment of ν .

To prove the second assertion, we write

$$\psi(\xi) = \psi^{(\alpha)}(\xi) + \xi \cdot A\xi + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) (\nu - \nu^{(\alpha)})(dy).$$

Since $0 \leq \xi \cdot A\xi \leq \|A\| |\xi|^2$, we only need to show that the last member above is of order $o(|\xi|^\alpha)$. We have

$$\left| \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) (\nu - \nu^{(\alpha)})(dy) \right| \leq \frac{1}{2} \int_{|y| \leq 1/|\xi|} (|\xi||y|)^2 |\nu - \nu^{(\alpha)}|(dy) + 2 \int_{|y| > 1/|\xi|} |\nu - \nu^{(\alpha)}|(dy),$$

and the statement follows from the assumption. \square

In what follows we will often use the following notation. If $X = (X_t)_{t \geq 0}$ is a symmetric Lévy process with characteristic exponent ψ as in (1.2), then we write

$$X = X^A + X^\nu \quad \text{and} \quad \psi(\xi) = \psi_A(\xi) + \psi_\nu(\xi),$$

where $X^A = (X_t^A)_{t \geq 0}$ is the Gaussian part determined by the Lévy-Khintchine exponent $\psi_A(\xi) = \xi \cdot A\xi$, and $X^\nu = (X_t^\nu)_{t \geq 0}$ is the jump part with the exponent $\psi_\nu(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot z)) \nu(dz)$.

The following fact on the tails of jump Lévy processes with nondegenerate Gaussian component will also be needed below. It states that one can add a sufficiently regular diffusion process to a purely jump Lévy process without spoiling the assumption **(U)**.

Proposition 5.3. *Let X be a Lévy process determined by the Lévy-Khintchine exponent ψ as in (1.2) with Gaussian coefficient $A = (a_{ij})_{1 \leq i, j \leq d}$ and Lévy measure ν . Moreover, suppose that $\inf_{|\xi|=1} \xi \cdot A\xi > 0$. If the process X^ν satisfies the assumption (U) with $\gamma > 0$, profile F and constants C_2, r_0, t_1 , then the entire process X also satisfies a version of (U). More precisely, there are constants $\tilde{C}_2 \geq C_2$ and $C_4 \in (0, 1]$ such that*

$$\mathbf{P}_0(|X_t| \geq r) \leq \tilde{C}_2 t^\gamma (F(C_4 r) \vee e^{-C_4 r}), \quad r \geq 2(r_0 \vee 2t), \quad t \geq t_1.$$

In particular, if $F(C_4 r) \geq e^{-C_4 r}$ for $r \geq 2r_0$, then X satisfies the assumption (U) with \tilde{C}_2 , the same γ and the profile $\tilde{F}(r) = F(C_4 r)$. If $F(C_4 r) < e^{-C_4 r}$ for $r \geq 2r_0$, then the same is true with \tilde{C}_2 , γ and $\tilde{F}(r) = e^{-C_4 r}$.

Proof. For $t > t_1$ and $r \geq 4t$, we may write

$$\begin{aligned} \mathbf{P}_0(|X_t| \geq r) &= \mathbf{P}_0(|X_t^\nu + X_t^A| \geq r) \\ &\leq \mathbf{P}_0\left(|X_t^\nu| \geq \frac{r}{2}\right) + \mathbf{P}_0\left(|X_t^A| \geq \frac{r}{2}\right). \end{aligned}$$

The first part is estimated using (U). To take care of the Gaussian part, note that under the assumption $\inf_{|\xi|=1} \xi \cdot A\xi > 0$ the transition densities $p_A(t, x, y)$ of the corresponding diffusion process X^A exist and enjoy the Gaussian upper estimates:

$$p_A(t, x, y) \leq \frac{c_1}{t^{d/2}} e^{-c_2 \frac{|x-y|^2}{t}}.$$

In particular, taking into account that $r > 4t \geq 4t_1$,

$$\mathbf{P}_0\left(|X_t^A| \geq \frac{r}{2}\right) = \int_{B(0, r/2)^c} p_A(t, 0, y) dy \leq \int_{B(0, r/2)^c} \frac{c_1}{t^{d/2}} e^{-c_2 \frac{|y|^2}{t}} dy \leq c_3 e^{-c_2 r},$$

so that

$$\begin{aligned} \mathbf{P}_0(|X_t| \geq r) &\leq C_2 t^\gamma (F(r/2) \vee e^{-r/2}) + c_3 e^{-c_2 r} \\ &\leq c_4 t^\gamma (F(c_5 r) \vee e^{-c_5 r}). \end{aligned}$$

for some positive constants $c_1 - c_5$. The proof is complete. \square

In the sequel, we will also need the following general lower estimate for the function G defined in (4.4). Recall that for every $R > 0$ we have denoted $U_R = (-R, R)^d$. Below we will also write $B(R)$ for $B(0, R)$.

Proposition 5.4. *Let X be a Lévy process with Lévy-Khintchine exponent ψ given by (1.2) and such that $e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^d)$, $t > 0$. Suppose there exist $C_5 > 0$ and $r_0 > 0$ such that for every Borel set $E \subset \mathbb{R}^d$*

$$(5.7) \quad \nu(E \cap \{x : 0 < |x| \leq 2r_0\}) \geq C_5 |E \cap \{x : 0 < |x| \leq 2r_0\}|.$$

Then for every $t > 0$, $R > 4r_0$, and $2r_0 < |y| < \frac{R}{2}$ one has

$$(5.8) \quad p^{U_R}(t, 0, y) \geq \eta(t, r_0) \inf_{|z| \leq \frac{R}{2} + r_0} \nu(B(z, r_0)).$$

where

$$(5.9) \quad \eta(t, r) = \int_0^t \mathbf{P}_0(\tau_{B(r)} > s) \Lambda(t-s, r) \min\left(1, \Lambda(t-s, r) \left|B\left(\frac{r}{8}\right)\right|\right) ds.$$

with

$$\Lambda(t, r) = C_5 \int_0^{t/2} \mathbf{P}_0(\tau_{B(r/8)} > u) \mathbf{P}_0(\tau_{B(r/8)} > t/2 - u) du.$$

In particular,

$$(5.10) \quad G(4r_0, R) \geq \eta(1, r_0) \inf_{|y| \leq \frac{R}{2} + r_0} \nu(B(y, r_0)).$$

Proof. Let $r_0 > 0$ be as in the assumptions and let $R > 4r_0$ be fixed. One can check by using the strong Markov property that for any $2r_0 < |y| < R/2$ and any $t > 0$

$$(5.11) \quad p^{U_R}(t, 0, y) = \mathbf{E}_0[\tau_{B(r_0)} < t; p^{U_R}(t - \tau_{B(r_0)}, X_{\tau_{B(r_0)}}, y)].$$

Set $\nu(x, \cdot) := \nu(\cdot - x)$. Using (5.11) and the Ikeda-Watanabe formula [14, Theorem 1], we can write for such y and t

$$\begin{aligned} p^{U_R}(t, 0, y) &\geq \mathbf{E}_0 \left[X_{\tau_{B(r_0)}} \in B(y, r_0), \tau_{B(r_0)} < t; p^{U_R}(t - \tau_{B(r_0)}, X_{\tau_{B(r_0)}}, y) \right] \\ &= \int_0^t \int_{B(r_0)} p^{B(r_0)}(s, 0, z) \int_{B(y, r_0)} p^{U_R}(t - s, w, y) \nu(z, dw) dz ds \\ &\geq \int_0^t \int_{B(r_0)} p^{B(r_0)}(s, 0, z) \inf_{w \in B(y, r_0)} p^{B(y, 2r_0)}(t - s, y, w) \nu(z, B(y, r_0)) dz ds \\ &\geq \left[\int_0^t \mathbf{P}_0(\tau_{B(r_0)} > s) \inf_{|x| \leq r_0} p^{B(2r_0)}(t - s, 0, x) ds \right] \inf_{|z| \leq \frac{R}{2} + r_0} \nu(B(z, r_0)). \end{aligned}$$

To complete the proof, we need to estimate the kernel $p^{B(2r_0)}(t', 0, x)$ for every $t' \in (0, t]$ and $|x| \leq r_0$. Let first $r_0/2 < |x| \leq r_0$. By following through with the argument above and using (5.7), we have

$$\begin{aligned} p^{B(2r_0)}(t', 0, x) &\geq \mathbf{E}_0 \left[X_{\tau_{B(r_0/4)}} \in B(x, r_0/4), \tau_{B(r_0/4)} < t'; p^{B(2r_0)}(t' - \tau_{B(r_0/4)}, X_{\tau_{B(r_0/4)}}, x) \right] \\ &= \int_0^{t'} \int_{B(r_0/4)} p^{B(r_0/4)}(u, 0, z) \int_{B(x, r_0/4)} p^{B(2r_0)}(t' - u, w, x) \nu(z, dw) dz du \\ (5.12) \quad &\geq C_5 \int_0^{t'} \mathbf{P}_0(\tau_{B(r_0/4)} > u) \int_{B(x, r_0/4)} p^{B(x, r_0/4)}(t' - u, x, w) dw du \\ &= C_5 \int_0^{t'} \mathbf{P}_0(\tau_{B(r_0/4)} > u) \mathbf{P}_0(\tau_{B(r_0/4)} > t' - u) du. \end{aligned}$$

In the case when $|x| \leq r_0/2$, use first the Chapman-Kolmogorov identity and then (5.12) to get

$$\begin{aligned} p^{B(2r_0)}(t', 0, x) &\geq \int_{r_0/2 < |z| \leq r_0} p^{B(2r_0)}(t'/2, 0, z) p^{B(2r_0)}(t'/2, z, x) dz \\ &\geq C_5 \left[\int_0^{t'/2} \mathbf{P}_0(\tau_{B(r_0/4)} > u) \mathbf{P}_0(\tau_{B(r_0/4)} > t'/2 - u) du \right] \int_{\frac{3r_0}{8} < |x-z| \leq \frac{3r_0}{4}} p^{B(x, \frac{3r_0}{2})}(t'/2, x, z) dz \\ &\geq C_5 \left[\int_0^{t'/2} \mathbf{P}_0(\tau_{B(r_0/4)} > u) \mathbf{P}_0(\tau_{B(r_0/4)} > t'/2 - u) du \right] \left| B\left(\frac{r_0}{8}\right) \right| \inf_{\frac{3r_0}{8} < |z| \leq \frac{3r_0}{4}} p^{B(\frac{3r_0}{2})}(t'/2, 0, z). \end{aligned}$$

Here we have used the fact that the set $(\overline{B}(r_0) \setminus \overline{B}(r_0/2)) \cap (\overline{B}(x, 3r_0/4) \setminus \overline{B}(x, 3r_0/8))$ always contains a ball of radius $r_0/8$ (in the last line we first restricted the integration to this ball and then we took the infimum). Observe that the last infimum can be estimated exactly in the same way as in (5.12). We thus have

$$\begin{aligned} p^{B(\frac{3r_0}{2})}(t'/2, 0, z) &\geq \mathbf{E}_0 \left[X_{\tau_{B(r_0/8)}} \in B(z, r_0/8), \tau_{B(r_0/8)} < t'/2; p^{B(\frac{3r_0}{2})}(t'/2 - \tau_{B(r_0/8)}, X_{\tau_{B(r_0/8)}}, z) \right] \\ &\geq C_5 \int_0^{t'/2} \mathbf{P}_0(\tau_{B(r_0/8)} > u) \mathbf{P}_0(\tau_{B(r_0/8)} > t'/2 - u) du, \end{aligned}$$

as long as $\frac{3r_0}{8} < |z| \leq \frac{3r_0}{4}$. In consequence, for $0 < s < t$ we have

$$\inf_{|x| \leq r_0} p^{B(2r_0)}(t-s, 0, x) \geq \Lambda(1 \wedge (\Lambda|B(0, r_0/8)|)),$$

where we have denoted $\Lambda = \Lambda(t', r_0) := C_5 \int_0^{t'/2} \mathbf{P}_0(\tau_{B(r_0/8)} > u) \mathbf{P}_0(\tau_{B(r_0/8)} > t'/2 - u) \, du$.

Finally,

$$p^{U_R}(t, 0, y) \geq \eta(t, r_0) \inf_{|z| \leq \frac{R}{2} + r_0} \nu(B(z, r_0)), \quad 2r_0 < |y| < \frac{R}{2},$$

with $\eta(t, r)$ defined by (5.9). The Proposition follows. \square

In the sequel we will make use of the following symmetrization of the exponent ψ . Denote

$$(5.13) \quad \Psi(r) = \sup_{|\xi| \leq r} \psi(\xi), \quad r > 0.$$

It follows from a combination of [34, Remark 4.8] and [30, Section 3] that there exist constants $C_6, C_7 > 0$, independent of the process (i.e. of A and ν), such that

$$(5.14) \quad C_6 H\left(\frac{1}{r}\right) \leq \Psi(r) \leq C_7 H\left(\frac{1}{r}\right), \quad r > 0, \quad \text{where} \quad H(r) = \frac{\|A\|}{r^2} + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 \wedge \frac{|y|^2}{r^2}\right) \nu(dy)$$

($\|A\|$ denotes the operator norm of a square matrix A). A direct proof of this estimate with explicit constants can be found in [12, Lemma 6]. It immediately follows from the definition that H is non-increasing and that the doubling property $H(r) \leq 4H(2r)$, $r > 0$, holds. In particular, $\Psi(2r) \leq 4C_6^{-1}C_7\Psi(r)$, for all $r > 0$. Also, by (5.14) we get that $\nu(B(0, r)^c) \leq C_6^{-1}\Psi(1/r)$, $r > 0$.

5.1. Processes with polynomially decaying Lévy measures

In this subsection we show how our general results translate to the case when the Lévy measure is polynomially decaying at infinity. We now give versions of Theorems 3.1 and 4.1 specialized to this case. Recall that for a symmetric α -stable process with Lévy-Khinchine exponent $\psi^{(\alpha)}$, by $\lambda_1^{(\alpha)}(U)$ we denote the principal Dirichlet eigenvalue of a set U , and by $\lambda_{(\alpha)}$ – the infimum of $\lambda_1^{(\alpha)}$ over all open sets of unit measure.

We first consider the class of Lévy processes that are close to non-Gaussian symmetric stable processes in the sense of the condition (C). As we will see later (Lemma 5.1), when the Lévy measure of such process has a density comparable with a nonincreasing function, then its decay at infinity is necessarily stable-like.

Theorem 5.2. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2), with Gaussian coefficient A and Lévy measure ν such that (C) with $\alpha \in (0, 2)$ and (5.1) hold, i.e.*

- (i) $\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha)$ as $|\xi| \rightarrow 0$ for some $\alpha \in (0, 2)$, where $\psi^{(\alpha)}(\xi)$ is defined in (1.5),

and

- (ii) $\lim_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{(\log|\xi|)^2} = \infty$.

Further, let V^ω be a Poissonian potential with bounded, compactly supported, nonnegative and nonidentically zero profile W . Then the following hold.

- (a) For any fixed $x \in \mathbb{R}^d$ one has

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \leq -\alpha \left(\frac{2}{d+2\alpha} \right)^{\frac{d}{d+\alpha}} \left(\frac{\rho}{d} \right)^{\frac{\alpha}{d+\alpha}} (\lambda_{(\alpha)})^{\frac{d}{d+\alpha}}, \quad \mathbb{Q} - a.s.$$

(b) If there exist $C_8, C_9, r_0 > 0$ such that for every Borel set $E \subset \mathbb{R}^d$

$$\nu(E \cap \{x : 0 < |x| \leq r_0\}) \geq C_8 |E \cap \{x : 0 < |x| \leq r_0\}|,$$

and

$$\nu(B(x, r_0)) \geq C_9 |x|^{-d-\alpha}, \quad \text{for } |x| \geq r_0,$$

then for any fixed $x \in \mathbb{R}^d$ one has

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \geq - \left(2\alpha + \frac{9d}{2}\right) \left(\frac{2}{d+2\alpha}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho\omega_d}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_1^{(\alpha)}(B(0, 1))\right)^{\frac{d}{d+\alpha}}, \quad \mathbb{Q} - a.s.$$

Proof. (a) To prove the upper bound we will apply our general Theorem 3.1. First, we verify condition (U). From [30, Section 3 and (3.2)] (see also [34, Lemma 4.1]) and (5.14) combined with the basic asymptotic assumption (i) we get

$$\mathbf{P}_0(|X_t| > r) \leq \mathbf{P}_0\left(\sup_{s \in (0, t]} |X_s| > r\right) \leq c_1 t H(r) \leq c_2 t \Psi\left(\frac{1}{r}\right) \leq c_3 \frac{t}{r^\alpha},$$

for every $t > 0$ and sufficiently large $r > 0$, with some constants $c_1, c_2, c_3 > 0$. Thus (U) holds with $F(r) = r^{-\alpha}$ and $\gamma = 1$.

From Theorem 5.1 we also see that (3.8) is satisfied with $\kappa_0 = \rho(\lambda_{(\alpha)})^{d/\alpha}$. To manage the correction terms appearing in the statement of Theorem 3.1, observe that for given $\kappa > 0$ (cf. (3.5), (3.6))

$$f_{F, \alpha, \kappa}(r) = \left(\alpha + \frac{d}{2}\right) \left(\frac{d}{\kappa}\right)^{\alpha/d} (\log r)^{\frac{\alpha+d}{d}} \quad \text{for sufficiently large } r,$$

so that

$$h_{F, \alpha, \kappa}(t) = \exp \left\{ \left(\frac{\kappa}{d}\right)^{\frac{\alpha}{d+\alpha}} \left(\frac{2}{2\alpha+d}\right)^{\frac{d}{d+\alpha}} t^{\frac{d}{d+\alpha}} \right\}, \quad \text{for large } t,$$

and

$$g(t) = g_{F, \alpha, \kappa}(t) = \left(\frac{2}{2\alpha+d} \left(\frac{\kappa}{d}\right)^{\frac{\alpha}{d}}\right)^{-\frac{\alpha}{d+\alpha}} t^{\frac{d}{d+\alpha}}.$$

By inspection we check that

$$(5.15) \quad \frac{\log h_{F, \alpha, \kappa}(t)}{g_{F, \alpha, \kappa}(t)} = \left(\frac{\kappa}{d}\right)^{\frac{\alpha}{d}} \left(\frac{2}{2\alpha+d}\right).$$

Using this observation with $\kappa = \kappa_0$, from (3.8) and (5.15) we get that \mathbb{Q} -almost surely

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \leq -\frac{2\alpha}{2\alpha+d} \left(\frac{\kappa_0}{d}\right)^{\frac{\alpha}{d}} = -\frac{2\alpha}{2\alpha+d} \left(\frac{\rho}{d}\right)^{\frac{\alpha}{d}} \lambda_{(\alpha)},$$

or

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \leq -\alpha \left(\frac{2}{2\alpha+d}\right)^{\frac{d}{d+\alpha}} \left(\frac{\kappa_0}{d}\right)^{\frac{\alpha}{d+\alpha}} = -\alpha \left(\frac{2}{2\alpha+d}\right)^{\frac{d}{d+\alpha}} \left(\frac{\rho}{d}\right)^{\frac{\alpha}{d+\alpha}} (\lambda_{(\alpha)})^{\frac{d}{d+\alpha}}.$$

(b) Proposition 5.1 asserts that the assumptions of Theorem 4.1 are satisfied with any $K > 1$. To match the asymptotic profile from the upper bound, we take again $F(r) = r^{-\alpha}$, and $R_0 = 2r_0$. We shall first obtain the lower bound with any given $\kappa > 0$, and at the end we will choose a suitable κ . We first check the assumptions of Corollary 4.1. Due to Proposition 5.4 and the lower bound on ν in (b) we see that $G(2r_0, R) \geq c_4 R^{-d-\alpha}$ for large R , with $c_4 > 0$, therefore

$$(5.16) \quad \limsup_{r \rightarrow \infty} \frac{\left| \log G\left(2r_0, \frac{2\sqrt{d}r}{(\log r)^{\frac{2}{d+2}}}\right) \right|}{r \wedge |\log F(r)| + (d/2) \log r} \leq \frac{d+\alpha}{d/2+\alpha},$$

and we can take $Q_1 = \alpha$, $Q_2 = \frac{\alpha+d}{\alpha+d/2}$ in (4.14). This gives that, for any fixed $\kappa > 0$ and $x \in \mathbb{R}^d$, \mathbb{Q} -almost surely,

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \geq -K \left(\frac{\omega_d \rho}{d} \right)^{\alpha/d} \lambda_1^{(\alpha)}(B(0, 1)) - \left(\frac{\kappa}{d} \right)^{\alpha/d} \left(\frac{2\alpha + 8d}{2\alpha + d} \right),$$

i.e.

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \geq - \frac{K \lambda_1^{(\alpha)}(B(0, 1)) \left(\frac{\omega_d \rho}{d} \right)^{\frac{\alpha}{d}} + (\alpha + 4d) \left(\frac{\kappa}{d} \right)^{\frac{\alpha}{d}} \frac{2}{2\alpha + d}}{\left(\left(\frac{\kappa}{d} \right)^{\frac{\alpha}{d}} \frac{2}{2\alpha + d} \right)^{\frac{\alpha}{d+\alpha}}}.$$

Letting $K \downarrow 1$ through rationals, we get this statement with $K = 1$. To match the upper bound, we take $\kappa = \kappa_0 = \rho \omega_d \left(\lambda_1^{(\alpha)}(B(0, 1)) \right)^{\frac{d}{\alpha}}$ i.e. the value corresponding to that in the upper bound. We easily check that with this choice of κ we get

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \geq - \left(2\alpha + \frac{9d}{2} \right) \left(\frac{2}{2\alpha + d} \right)^{\frac{d}{d+\alpha}} \left(\frac{\rho \omega_d}{d} \right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_1^{(\alpha)}(B(0, 1)) \right)^{\frac{d}{d+\alpha}}, \quad \mathbb{Q} - \text{a.s.}$$

The proof is complete. \square

Remark 5.1.

- (1) When $\inf_{|\xi|=1} \psi_A(\xi) > 0$, then (ii) automatically holds and needs not to be assumed a priori.
- (2) If $\nu(dx) = \nu(x)dx$ and there exists a nonincreasing profile function $g : (0, \infty) \rightarrow (0, \infty)$ such that $\nu(x) \asymp g(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$, then condition (i) implies that

$$(5.17) \quad g(r) \asymp \Psi(1/r) r^{-d} \asymp r^{-d-\alpha}, \quad \text{for all } r \geq 1,$$

where Ψ is the symmetrization of ψ defined in (5.13). In particular, the assumption in part (b) of the theorem automatically holds and can be omitted. A short proof of (5.17) is given in Lemma 5.1 below.

- (3) When the process $X^{(\alpha)}$ is isotropic, then our upper and lower bound differ by just a multiplicative constant (see the comment following Theorem 5.1). However, we were not able to get the almost sure convergence in this theorem, even for isotropic processes. We do not know whether it is a flaw of the method, or if it is an intrinsic feature of the functional, signaling the existence of some ‘small deviations’ phenomenon here. The detailed study of this case is the purpose of an ongoing project.
- (4) In the final part of the proof above we could do better: choose $\kappa = a\kappa_0$ and then optimize over $a > 0$. This gives:

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+\alpha}}} \geq -(\alpha + 4d)^{\frac{\alpha}{d+\alpha}} \left(\left(\frac{d}{\alpha} \right)^{\frac{\alpha}{d+\alpha}} + \left(\frac{\alpha}{d} \right)^{\frac{d}{d+\alpha}} \right) \left(\frac{\rho \omega_d}{d} \right)^{\frac{\alpha}{d+\alpha}} \left(\lambda_1^{(\alpha)}(B(0, 1)) \right)^{\frac{d}{d+\alpha}}, \quad \mathbb{Q} - \text{a.s.}$$

This is the best lower bound that can be derived from Theorem 4.1.

Lemma 5.1. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2), with Gaussian coefficient $A \equiv 0$ and Lévy measure $\nu(dx) = \nu(x)dx$, and such that there exists a nonincreasing profile function $g : (0, \infty) \rightarrow (0, \infty)$ for which $\nu(x) \asymp g(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$. Then condition (i) of Theorem 5.2 implies (5.17).*

Proof. First note that by [21, Lemma 5 (a)] and (i) of Theorem 5.2, we have $\Psi(|x|) \asymp \psi(x) \asymp |x|^\alpha$, whenever $|x| \leq 1$. As in this case for $r \geq 1$ we have (see (5.14))

$$\frac{1}{r^\alpha} \asymp \Psi\left(\frac{1}{r}\right) \asymp H(r) = \int_{\mathbb{R}^d \setminus \{0\}} \left(1 \wedge \frac{|y|^2}{r^2} \right) \nu(dy),$$

so that there are constants $c_1, c_2 > 0$ for which

$$(5.18) \quad \frac{c_1}{r^\alpha} \leq \frac{1}{r^2} \int_{|y| < r} |y|^2 g(|y|) dy + \int_{|y| \geq r} g(|y|) dy \leq \frac{c_2}{r^\alpha}, \quad r \geq 1.$$

By the monotonicity of g , for $r \geq 1$ we get

$$\frac{c_2}{r^\alpha} \geq \frac{1}{r^2} \int_{|y| < r} |y|^2 g(|y|) dy \geq c_3 g(r) r^d.$$

To show the opposite inequality, observe that for any $u \in (0, 1)$ and $s > 1$ from the monotonicity of g and the upper bound in (5.18), we may write

$$\int_{|y| \geq r} g(|y|) dy = \int_{r \leq |y| < sr} g(|y|) dy + \int_{|y| \geq sr} g(|y|) dy \leq c_4 (s^d - 1) g(r) r^d + \frac{c_2}{s^\alpha} \frac{1}{r^\alpha}, \quad r \geq 1,$$

for some $c_4 > 0$. Similarly,

$$\begin{aligned} \frac{1}{r^2} \int_{|y| < r} |y|^2 g(|y|) dy &= u^2 \frac{1}{(ur)^2} \int_{|y| < ur} |y|^2 g(|y|) dy + \frac{1}{r^2} \int_{ur \leq |y| < r} |y|^2 g(|y|) dy \\ &\leq \frac{c_2 u^{2-\alpha}}{r^\alpha} + c_4 (1 - u^d) g(ur) r^d, \quad r \geq \frac{1}{u}. \end{aligned}$$

Adding these two estimates, from (5.18) we obtain, using also the monotonicity of g :

$$\frac{c_1}{r^\alpha} \leq \frac{c_2 u^{2-\alpha}}{r^\alpha} + \frac{c_2}{s^\alpha r^\alpha} + c_4 (s^d - u^d) g(ur) r^d.$$

Choosing $u_0 \in (0, 1)$ so small and $s_0 > 1$ so large that $c_2(u_0^{2-\alpha} + s_0^{-\alpha}) \leq c_1/2$, we finally obtain

$$\frac{c_1}{2r^\alpha} \leq c_4 (s_0^d - u_0^d) g(u_0 r) r^d, \quad r \geq \frac{1}{u_0},$$

and, as a consequence, $r^{-d-\alpha} \leq \frac{2c_4(s_0^d - u_0^d)}{c_1 u_0^{d-\alpha}} g(r)$, for every $r \geq 1$, which is the claimed inequality. This completes the proof. \square

We now illustrate our Theorem 5.2 with several examples.

Example 5.1. (Absolutely continuous perturbations of isotropic stable processes)

- (1) *Non-Gaussian isotropic stable processes.* Let $\psi(\xi) = |\xi|^\delta$, with $\delta \in (0, 2)$. In this case $\nu(dx) = C_{d,\delta} |x|^{-d-\delta} dx$ and we have to take $\alpha = \delta$ and $\psi^{(\alpha)} = \psi$. In particular, the assumptions of Theorem 5.2 hold.
- (2) *Mixture of isotropic stable processes (possibly with Brownian component).* Let $\psi(\xi) = a_0 |\xi|^2 + \sum_{i=1}^n a_i |\xi|^{\alpha_i}$, $n \in \mathbb{N}$, $a_0 \geq 0$, and $a_i > 0$, $\alpha_i \in (0, 2)$, for $i = 1, \dots, n$. Then the assumptions of Theorem 5.2 are satisfied with $\alpha = \alpha_{\min} := \min_i \alpha_i$ and $\psi^{(\alpha)}(\xi) = a_{i_{\min}} |\xi|^{\alpha_{i_{\min}}}$.
- (3) *Isotropic geometric stable process with Gaussian component.* Let $\psi(\xi) = \xi \cdot A\xi + \log(1 + |\xi|^\delta)$, with A such that $\inf_{|\xi|=1} \xi \cdot A\xi > 0$ and $\delta \in (0, 2)$. Again, Theorem 5.2 applies with $\alpha = \delta$ and $\psi^{(\alpha)}(\xi) = |\xi|^\alpha$.

Example 5.2. (More general perturbations of symmetric stable processes)

Let $\delta \in (0, 2)$ and let n be a symmetric finite measure on the unit sphere S^{d-1} such that

$$n(B(\theta, r) \cap S^{d-1}) \geq c_0 r^{d-1}, \quad \theta \in S^{d-1}, \quad r \in (0, 1/2],$$

for some constant $c_0 > 0$. Denote the corresponding stable Lévy measure by $\nu^{(\delta)}(dr d\theta) = n(d\theta) r^{-1-\delta} dr$. Note that we do not impose similar growth condition on n from above, which means that $\nu^{(\delta)}$ is not necessarily absolutely continuous with respect to the Lebesgue measure. Furthermore, let ν_∞ be a (non-necessarily

infinite) measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$(5.19) \quad \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge (r|z|)^2) \nu_\infty(dz) = o(r^\delta) \quad \text{as } r \rightarrow 0,$$

and consider a symmetric Lévy process with Lévy-Khinchine exponent ψ as in (1.2) with arbitrary diffusion matrix A and Lévy measure $\nu = \nu^{(\delta)} + \nu_\infty$. Then the assumptions of Theorem 5.2 hold with $\alpha = \delta$ and $\psi^{(\alpha)}(\xi) = \int_0^\infty \int_{S^{d-1}} (1 - \cos(\xi \cdot r\theta)) n(d\theta) \frac{dr}{r^{\alpha+1}}$. Typical examples of measures ν_∞ satisfying (5.19) are as follows.

- (1) *Other stable Lévy measures:* $\nu_\infty := \nu^{(\tilde{\delta})}(dr d\theta) = \tilde{n}(d\theta) r^{-1-\tilde{\delta}} dr$, where $\tilde{\delta} \in (\delta, 2)$ and \tilde{n} is a symmetric finite measure on S^{d-1} .
- (2) *Product Lévy measures with profiles with sufficiently fast decay at infinity:* $\nu_\infty := \tilde{n}(d\theta) f(r) dr$, where \tilde{n} is a symmetric finite measure on S^{d-1} and $f : (0, \infty) \rightarrow [0, \infty)$ is a function such that $\int_0^\infty s^2 f(s) ds < \infty$.
- (3) *Purely discrete Lévy measures.* Let $\{v_k : k = 1, \dots, k_0\}$ be a family of $k_0 \in \mathbb{N}$ vectors in \mathbb{R}^d . For $q > 0$ denote

$$A_q = \left\{ x \in \mathbb{R}^d : x = 2^{qn} v_k, \text{ where } n \in \mathbb{Z}, k = 1, \dots, k_0 \right\}$$

and

$$f(s) := \mathbf{1}_{[0,1]}(s) \cdot s^{-\theta/q} + \mathbf{1}_{(1,\infty)}(s) \cdot s^{-\beta/q}, \quad s > 0,$$

with $\theta \in (0, 2q)$ and $\beta > 2q$. Then one can take

$$\nu_\infty(dy) := \int_{\mathbb{R}^d} f(|y|) \delta_{A_q}(dy) = \sum_{y \in A_q} f(|y|).$$

We now turn to the class of processes with Lévy measures that have second moment finite. In this case, for a given Gaussian matrix $A = (a_{ij})_{1 \leq i, j \leq d}$, the coefficients

$$(5.20) \quad \tilde{A} = (\tilde{a}_{ij})_{1 \leq i, j \leq d}, \quad \text{with } \tilde{a}_{ij} = a_{ij} + \frac{1}{2} \int_{\mathbb{R}^d} y_i y_j \nu(y) dy,$$

are well defined (see Proposition 5.2 (i)). In what follows, by $\lambda_1^{(2)}(U)$ we will always denote the principal eigenvalue of the diffusion process with characteristic exponent $\psi^{(2)}(\xi) = \xi \cdot \tilde{A} \xi$, killed on leaving an open bounded set $U \subset \mathbb{R}^d$. Also, $\lambda_{(2)}$ denotes the infimum of $\lambda_1^{(2)}(U)$ over all open sets U of unit measure (recall the comment after Theorem 5.1).

We now discuss the case of Lévy measures with polynomial tails whose decay at infinity is faster than stable. To avoid some technical difficulties and for more clarity, in the theorem below we restrict our attention to the absolutely continuous case (cf. Example 5.5 (2)).

Theorem 5.3. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2), with defining parameters $A = (a_{ij})_{1 \leq i, j \leq d}$ and $\nu(dx) = \nu(x) dx$. Assume that*

- (i) *either $\inf_{|\xi|=1} \psi_A(\xi) > 0$, $\liminf_{|\xi| \rightarrow \infty} \frac{\psi_\nu(\xi)}{\log|\xi|} > 0$ or $\psi_A(\xi) \equiv 0$, $\lim_{|\xi| \rightarrow \infty} \frac{\psi_\nu(\xi)}{(\log|\xi|)^2} = \infty$,*
- (ii) *there exist $C_{10} \geq C_{11} > 0$ and $\delta_1 \geq \delta_2 > 2$ such that*

$$C_{11} \left(\frac{1}{1 + |x|^{d+\delta_1}} \right) \leq \nu(x) \leq \frac{C_{10}}{|x|^{d+\delta_2}}, \quad x \in \mathbb{R}^d.$$

Moreover, let V^ω be a Poissonian potential with bounded, compactly supported, nonnegative and nonidentically zero profile W . Then, for any fixed $x \in \mathbb{R}^d$, one has

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+2}}} \leq -\delta_2 \left(\frac{2}{2\delta_2 + d} \right)^{\frac{d}{d+2}} \left(\frac{\rho}{d} \right)^{\frac{2}{d+2}} (\lambda_{(2)})^{\frac{d}{d+2}}, \quad \mathbb{Q} - a.s.$$

and

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+2}}} \geq - \left(2\delta_1 + \frac{9d}{2} \right) \left(\frac{2}{2\delta_1 + d} \right)^{\frac{d}{d+2}} \left(\frac{\rho\omega_d}{d} \right)^{\frac{2}{d+2}} \left(\lambda_1^{(2)}(B(0, 1)) \right)^{\frac{d}{d+2}}, \quad \mathbb{Q} - a.s.$$

Proof. By Proposition 5.2 (i) not only the coefficients a_{ij} given by (5.20) are finite, but also one has

$$\psi(\xi) = \xi \cdot \tilde{A}\xi + o(|\xi|^2),$$

i.e. the basic asymptotic assumption **(C)** holds true with $\alpha = 2$. Also, (5.1) is satisfied in both cases of (i).

As usual, to establish the upper bound we apply our general Theorem 3.1. When $A \equiv 0$, then from Lemma 5.2 below we get $p(t, x) \leq c_1 t^{\delta_2/2} |x|^{-d-\delta_2}$, $x \in \mathbb{R}^d \setminus \{0\}$, $t \geq t_1$, for some $t_1 > 0$, so that **(U)** holds with $F(r) = r^{-\delta_2}$ and $\gamma = \delta_2/2$. When $\inf_{|\xi|=1} \xi \cdot A\xi > 0$, then the same is true by Proposition 5.3. Also, by Theorem 5.1, (3.8) holds true with $\kappa_0 = \rho(\lambda_{(2)})^{d/2}$ and the proof of the upper bound can be completed by following the argument in the proof of the upper bound in Theorem 5.2 above, with the function (for large r)

$$f_{F,2,\kappa}(r) = \left(\delta_2 + \frac{d}{2} \right) \left(\frac{d}{\kappa} \right)^{2/d} (\log r)^{\frac{2+d}{d}}.$$

To get the lower bound, it is enough to observe that by Proposition 5.4 and the bound on the density $\nu(x)$ we have $G(2, R) \geq c_2 R^{-d-\delta_1}$ for large R . Indeed, the rest of the proof follows the lines of the second part of the justification of the lower bound in Theorem 5.2 with profile function $F(r) = r^{-\delta_1}$. \square

To complete the proof of the above theorem we need to prove the following lemma.

Lemma 5.2. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2), with defining parameters $A \equiv 0$ and $\nu(dx) = \nu(x)dx$, such that $\liminf_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{\log |\xi|} > 0$. If there exist $C_{12} - C_{14} > 0$ and $\delta > 2$ such that $\nu(x) \leq C_{12}|x|^{-d-\delta}$ for $x \in \mathbb{R}^d \setminus \{0\}$ and $\nu(x) \geq C_{13}$ for $|x| \leq C_{14}$, then there exist $C_{15}, t_1 > 0$ such that*

$$p(t, x) \leq C_{15} \frac{t^{\frac{\delta}{2}}}{|x|^{d+\delta}}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t > t_1.$$

Proof. The assumption $\liminf_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{\log |\xi|} > 0$ immediately gives that $e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^d)$, for sufficiently large $t > 0$, and also that $\nu(\mathbb{R}^d \setminus \{0\}) = \infty$. Thus, by the Fourier inversion formula, $p(t, x)$ exist and are bounded for all t large enough. To find an upper bound on $p(t, x)$, we use [20, Theorem 1]. Its assumption (1) follows directly from the upper bound on the density $\nu(x)$ with the profile function $f(r) = r^{-d-\delta}$ and (2) can be directly derived from the monotonicity and the doubling property of such f . Indeed, for every $s, r > 0$ we may write

$$\int_{|y|>r} f\left(s \vee |y| - \frac{|y|}{2}\right) \nu(y) dy = \left(\int_{\substack{|y|>r \\ |y|\leq s}} + \int_{\substack{|y|>r \\ |y|>s}} \right) f\left(s \vee |y| - \frac{|y|}{2}\right) \nu(y) dy =: I_1 + I_2.$$

Since f is nonincreasing, both integrals I_1 and I_2 can be easily estimated by $f(s/2) \int_{|y|>r} \nu(y) dy$, which is smaller or equal to $c_1 f(s) \Psi(1/r)$, for all $s, r > 0$. Thus the assumption (2) holds true.

It remains to justify the last assumption (3). First note that by the upper estimate of the density $\nu(x)$ and Proposition 5.2 (i) one has

$$\psi(\xi) = \xi \cdot \tilde{A}\xi + o(|\xi|^2), \quad \text{with } \tilde{A} = (\tilde{a}_{ij})_{1 \leq i, j \leq d}, \quad \text{where } a_{ij} = \frac{1}{2} \int_{\mathbb{R}^d} y_i y_j \nu(y) dy.$$

Since $\nu(x)$ is separated from zero around the origin, this together with $\liminf_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{\log |\xi|} > 0$, imply that there exist $0 < r_1 < 1 < r_2$ such that

$$c_2 |\xi|^2 \leq \psi(\xi) \leq c_3 |\xi|^2 \quad \text{for } |\xi| \leq r_1 \quad \text{and} \quad \psi(\xi) \geq c_4 \log |\xi| \quad \text{for } |\xi| \geq r_2,$$

with some constants $c_2, \dots, c_4 > 0$. The above bounds and the fact that $\inf_{r_1 \leq |\xi| \leq r_2} \psi(\xi) > 0$ immediately give that for every $t > 0$ we may write

$$\int_{\mathbb{R}^d} e^{-t\psi(\xi)} |\xi| d\xi \leq \int_{|\xi| < r_1} e^{-c_2 t |\xi|^2} |\xi| d\xi + e^{-t \inf_{r_1 \leq |\xi| \leq r_2} \psi(\xi)} \int_{r_1 \leq |\xi| \leq r_2} |\xi| d\xi + \int_{|\xi| > r_2} e^{-c_4 t \log |\xi|} |\xi| d\xi.$$

When $t > \frac{d+1}{c_4}$, then the last integral is convergent, and, moreover, we get that there exists $c_5 > 0$ such that the estimate

$$\int_{\mathbb{R}^d} e^{-t\psi(\xi)} |\xi| d\xi \leq c_5 t^{-\frac{d+1}{2}}$$

holds for large $t > 0$. Since $\Psi^{-1}(1/t) \asymp t^{-1/2}$ for sufficiently large t , this in fact gives that there is $t_1 > 0$ such that the assumption (3) in [20, Theorem 1] holds with $T = [t_1, \infty)$. Consequently we have, with $h(t) \asymp t^{1/2}$ and $\gamma = d$, (due to the symmetry of the process the correction term $tb_{h(t)}$ is not present):

$$p(t, x) \leq c_6 \left(\frac{t}{|x|^{d+\delta}} + t^{-d/2} e^{-c_7 \frac{|x|}{t^{1/2}} \log(1+c_7 \frac{|x|}{t^{1/2}})} \right), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t \geq t_1.$$

We estimate the exponentially-logarithmic term above. If $|x| \geq t^{1/2}$, then

$$t^{-d/2} e^{-c_7 \frac{|x|}{t^{1/2}} \log(1+c_7 \frac{|x|}{t^{1/2}})} \leq c_8 t^{-d/2} (t^{1/2}/|x|)^{d+\delta} = c_8 t^{\delta/2} / |x|^{d+\delta},$$

for a constant $c_8 > 0$. On the other hand, for $|x| \leq t^{1/2}$ we simply have

$$t^{-d/2} e^{-c_7 \frac{|x|}{t^{1/2}} \log(1+c_7 \frac{|x|}{t^{1/2}})} \leq t^{-d/2} \leq t^{-d/2} (t^{1/2}/|x|)^{d+\delta} = t^{\delta/2} / |x|^{d+\delta}.$$

Altogether,

$$p(t, x) \leq c_9 \frac{t^{\frac{\delta}{2}}}{|x|^{d+\delta}}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t \geq t_1.$$

The statement follows. \square

Remark 5.2.

Similarly as before, in the final part of the proof of Theorem 5.3 above we could get a better bound. Indeed, by choosing $\kappa = a\kappa_0$ and optimizing over $a > 0$, one has

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{d}{d+2}}} \geq -(\delta_1 + 4d)^{\frac{2}{d+2}} \left(\left(\frac{d}{2} \right)^{\frac{2}{d+2}} + \left(\frac{2}{d} \right)^{\frac{d}{d+2}} \right) \left(\frac{\rho\omega_d}{d} \right)^{\frac{2}{d+2}} \left(\lambda_1^{(2)}(B(0, 1)) \right)^{\frac{d}{d+2}}, \quad \mathbb{Q} - \text{a.s.}$$

Example 5.3. (Layered stable process)

Our Theorem 5.3 above can be illustrated by considering a symmetric Lévy process with Lévy measure $\nu(dx) = \nu(x)dx$ such that $\nu(x) \asymp \mathbf{1}_{\{|x| \leq 1\}} |x|^{-d-\eta} + \mathbf{1}_{\{|x| > 1\}} |x|^{-d-\delta}$, $x \in \mathbb{R}^d \setminus \{0\}$, with $\eta \in (0, 2)$ and $\delta > 2$. Such processes are often called *layered stable*.

All the results above say that when $\psi(\xi) = \psi^{(\alpha)}(\xi) + o(|\xi|^\alpha)$, $\alpha \in (0, 2]$, $\xi \rightarrow 0$, and the Lévy measure decays polynomially at infinity, then the quenched rate of convergence of $u^\omega(t, x)$ as $t \rightarrow \infty$ is the same as its annealed rate of convergence (cf. (1.7)).

5.2. Processes with Lévy measures lighter than polynomial at infinity

We now show that when the tail of ν at infinity is lighter than polynomial, then the almost sure behaviour qualitatively changes and it is no longer true that it coincides with the annealed behaviour. In this case we are often able to get the convergence, i.e. to derive precisely the main term in the asymptotics.

First we discuss the example of a Lévy measure which decays at infinity faster than polynomially but still slower than stretched-exponentially.

Theorem 5.4. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2) with Gaussian coefficient $A = (a_{ij})_{1 \leq i, j \leq d}$ such that either $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A \xi > 0$, and a symmetric Lévy measure $\nu(dx) = \nu(x)dx$ such that there exist $\theta > 0$, $\beta > 1$, $\delta \in (0, 2)$ satisfying*

$$(5.21) \quad \nu(x) \asymp \mathbf{1}_{\{|x| \leq 1\}} |x|^{-d-\delta} + \mathbf{1}_{\{|x| > 1\}} e^{-\theta(\log |x|)^\beta}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Let V^ω be a Poissonian potential with bounded, compactly supported, nonnegative and nonidentically zero profile W . Then, for any fixed $x \in \mathbb{R}^d$, one has

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{\beta d}{2+d\beta}}} \leq -\theta^{\frac{2}{2+d\beta}} \left(\frac{\rho}{d}\right)^{\frac{2\beta}{2+d\beta}} (\lambda_{(2)})^{\frac{d\beta}{2+d\beta}}, \quad \mathbb{Q} - a.s.,$$

and

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{\beta d}{2+d\beta}}} \geq -2\theta^{\frac{2}{2+d\beta}} \left(\frac{\omega d \rho}{d}\right)^{\frac{2\beta}{2+d\beta}} (\lambda_1^{(2)}(B(0, 1)))^{\frac{d\beta}{2+d\beta}}, \quad \mathbb{Q} - a.s.,$$

where $\lambda_1^{(2)}(B(0, 1))$ and $\lambda_{(2)}$ correspond to the diffusion process with Gaussian matrix \tilde{A} given by (5.20).

Proof. First of all we observe that indeed by Proposition 5.2 (i) one has

$$\psi(\xi) = \xi \cdot \tilde{A} \xi + o(|\xi|^2),$$

i.e. the basic asymptotic assumption **(C)** holds true. By [21, Lemma 5 (a)], we have $\Psi_\nu(x) \asymp \psi_\nu(|x|)$, $x \in \mathbb{R}^d$, where Ψ_ν is the symmetrization of ψ_ν introduced in (5.13). From (5.14) we thus have

$$(5.22) \quad \psi(x) \geq \psi_\nu(x) \geq c_1 \int_{|y| > 1/|x|} \nu(y) dy \asymp |x|^\delta, \quad \text{for } |x| \text{ large enough.}$$

In particular, (5.1) holds true. We now address the upper bound and the lower bound separately.

THE UPPER BOUND. Similarly as before, we use our general Theorem 3.1. First we need to check that **(U)** holds true. When $A \equiv 0$, then it follows from Lemma 5.3 below that there exist $c_2 > 0$, $c_3 \in (0, 1/4]$, $r_0, t_1 > 0$ such that

$$(5.23) \quad p(t, x) \leq c_2 t e^{-\theta(\log(c_3|x|))^\beta}, \quad |x| > r_0 \vee t, \quad t \geq t_1,$$

which implies **(U)** with $\gamma = 1$ and $F(r) = e^{-\theta(\log(c_3 r))^\beta} r^d (\log c_3 r)^{-(\beta-1)}$ (clearly, such profile F is strictly increasing for $r > \tilde{r}_0$ with sufficiently large $\tilde{r}_0 \geq r_0$). When $\inf_{|\xi|=1} \xi \cdot A \xi > 0$, then we get from Proposition 5.3 that the same is true with the profile function $\tilde{F}(r) = e^{-\theta(\log(\tilde{c}_3 r))^\beta} r^d (\log \tilde{c}_3 r)^{-(\beta-1)}$ for some $\tilde{c}_3 \in (0, c_3)$ and same γ . Thanks to (5.22) and **(C)**, by Theorem 5.1 we also have

$$\lim_{\lambda \rightarrow 0} \lambda^{d/2} \log N^D(\lambda) = -\rho(\lambda_{(2)})^{d/2},$$

where $\lambda_{(2)}$ is determined by the variational formula (5.3) with $\lambda_1^{(2)}(G)$ corresponding to the diffusion process with exponent $\psi^{(2)}(\xi) = \xi \cdot \tilde{A} \xi$.

We are now in a position to derive the claimed upper bound. Indeed, with the preparation above, by our general Theorem 3.1 and Corollary 3.1, we get

$$(5.24) \quad \limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \leq -\left(\frac{\rho}{d}\right)^{2/d} \lambda_{(2)}, \quad \mathbb{Q} - a.s.,$$

with $g(t) = t/(\log h_{F, \alpha, \kappa_0}(t))^{2/d}$, where h_{F, α, κ_0} is the inverse function to f_{F, α, κ_0} given by (3.5) with $\alpha = 2$, $\kappa_0 = \rho(\lambda_{(2)})^{d/2}$ and $F(r) = e^{-\theta(\log(c_3 r))^\beta} r^d (\log c_3 r)^{-(\beta-1)}$ (as we will see below, here the concrete value of c_3 is irrelevant). By (3.5), since $\beta > 1$, we have

$$f_{F, 2, \kappa_0}(r) \approx \left(\theta(\log(c_3 r))^\beta + \frac{d}{2} \log r \right) \left(\frac{d \log r}{\kappa_0} \right)^{\frac{2}{d}} \approx \theta(\log r)^\beta \left(\frac{d \log r}{\kappa_0} \right)^{\frac{2}{d}}, \quad \text{for large } r.$$

Thus, by direct asymptotic calculations, we obtain that

$$\log h_{F,\alpha,\kappa_0}(t) \approx \left(\frac{1}{\theta}\right)^{\frac{d}{2+d\beta}} \left(\frac{\kappa_0}{d}\right)^{\frac{2}{2+d\beta}} t^{\frac{d}{2+d\beta}},$$

and consequently,

$$(5.25) \quad g(t) \approx \left(\frac{1}{\theta} \left(\frac{\kappa_0}{d}\right)^{\frac{2}{d}}\right)^{-\frac{2}{2+d\beta}} t^{\frac{\beta d}{2+d\beta}}.$$

Since $\kappa_0 = \rho(\lambda_{(2)})^{d/2}$, in light of (5.24), this gives

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{\frac{\beta d}{2+d\beta}}} \leq -\theta^{\frac{2}{2+d\beta}} \left(\frac{\rho}{d}\right)^{\frac{2\beta}{2+d\beta}} (\lambda_{(2)})^{\frac{d\beta}{2+d\beta}}. \quad \mathbb{Q} - \text{a.s.},$$

THE LOWER BOUND. First recall that at the beginning of the proof we verified the basic asymptotic assumption **(C)**. In view of Proposition 5.1 it gives that the assumptions of our general Theorem 4.1 (and Corollary 4.1 as well) are satisfied with any $K > 1$. To match the asymptotic profile from the upper bound, it is enough to take $F(r) = e^{-\theta(\log r)^\beta}$. Similarly as before, we first proceed with an arbitrary fixed $\kappa > 0$, and in the concluding part of the proof we will choose a suitable κ . Condition (5.7) of Proposition 5.4 is satisfied, so that we also have that there exists $c_4 > 0$ such that $G(1, R) \geq c_4 e^{-\theta(\log(R/2))^\beta}$ for sufficiently large R .

We now verify the assumptions of Corollary 4.1. First observe that one has $Q_1 = \infty$ in (4.11). Moreover, since

$$\limsup_{r \rightarrow \infty} \frac{\left| \log G\left(1, \frac{2\sqrt{d}r}{(\log r)^{\frac{2}{d}+2}}\right) \right|}{r \wedge |\log F(r)| + (d/2) \log r} \leq 1,$$

we also have $Q_2 = 1$ in (4.12). By (4.14), this yields that for any fixed $\kappa > 0$ and $x \in \mathbb{R}^d$

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{g(t)} \geq -K \left(\frac{\omega_d \rho}{d}\right)^{2/d} \lambda_1^{(2)}(B(0, 1)) - \left(\frac{\kappa}{d}\right)^{2/d}, \quad \mathbb{Q} - \text{a.s.}$$

Recall that here $\lambda_1^{(2)}(B(0, 1))$ corresponds to the diffusion process determined by $\psi^{(2)}(\xi) = \xi \cdot \tilde{A}\xi$ with \tilde{A} as in (5.20). In light of (5.25), passing to the limit $K \downarrow 1$ through rationals, we finally get

$$\liminf_{t \rightarrow \infty} \frac{u^\omega(t, x)}{t^{\frac{\beta d}{2+d\beta}}} \geq -\theta^{\frac{2}{2+d\beta}} \left(\frac{1}{d}\right)^{\frac{2\beta}{2+d\beta}} \left(\left(\omega_d \rho \kappa^{\frac{-2}{2+d\beta}}\right)^{2/d} \lambda_1^{(2)}(B(0, 1)) + \kappa^{\frac{2\beta}{2+d\beta}} \right), \quad \mathbb{Q} - \text{a.s.}$$

Again, to match the upper bound, we take $\kappa = \rho \omega_d \left(\lambda_1^{(2)}(B(0, 1))\right)^{\frac{d}{2}}$. With this choice of κ we conclude that

$$\liminf_{t \rightarrow \infty} \frac{u^\omega(t, x)}{t^{\frac{\beta d}{2+d\beta}}} \geq -2\theta^{\frac{2}{2+d\beta}} \left(\frac{\omega_d \rho}{d}\right)^{\frac{2\beta}{2+d\beta}} \left(\lambda_1^{(2)}(B(0, 1))\right)^{\frac{d\beta}{2+d\beta}}, \quad \mathbb{Q} - \text{a.s.},$$

which completes the proof. \square

We now justify the upper bound (5.23), which is one of the key steps in the proof of the above theorem.

Lemma 5.3. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2) with $A \equiv 0$ and $\nu(dx) = \nu(x)dx$ such that there exist $\theta > 0$, $\beta > 1$, $\delta \in (0, 2)$ satisfying*

$$(5.26) \quad \nu(-x) = \nu(x) \asymp \mathbf{1}_{\{|x| \leq 1\}} |x|^{-d-\delta} + \mathbf{1}_{\{|x| > 1\}} e^{-\theta(\log |x|)^\beta}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Then there exist $C_{16} > 0$, $r_0 > 0$ and $t_1 > 0$ such that

$$p(t, x) \leq C_{16} t e^{-\theta \left(\log \left(\frac{|x|}{4}\right)\right)^\beta}, \quad |x| > r_0 \vee t, \quad t \geq t_1.$$

Proof. We apply again [20, Theorem 1]. First observe that the assumption (1) of this theorem holds for $f(r) = \mathbf{1}_{\{r \leq 1\}} r^{-d-\delta} + \mathbf{1}_{\{r > 1\}} e^{-\theta(\log r)^\beta}$. We now justify its second assumption (2). As $\nu(x) \asymp f(|x|)$, we have to prove that there exists $c_1 > 0$ such that

$$(5.27) \quad \int_{|y|>r} f((s \vee |y|) - |y|/2) f(|y|) dy \leq c_1 f(s) \Psi(1/r), \quad s, r > 0,$$

where Ψ is given by (5.13) (see also (5.14) and comments after it). When $s \leq r$, then the integral on the left hand side can be directly estimated by $c_2 f(r) \int_{|y|>r} f(|y|/2) dy \leq c_2 f(s) \int_{|y|>r/2} f(|y|) dy \leq c_3 f(s) \Psi(2/r) \leq c_4 f(s) \Psi(1/r)$, which is the claimed bound. When $s > r$, then we split this integral into two integrals: over $|y| \geq s$ and $r < |y| < s$, respectively. In the first case, we can follow exactly the same argument as above. Consequently, we only need to estimate the second integral $I_{s,r} := \int_{r < |y| < s} f((s \vee |y|) - |y|/2) f(|y|) dy$. For $u_0 = e^{\beta-1}$ we have

$$(5.28) \quad \frac{(\log u_0)^{\beta-1}}{u_0} = \sup_{u \in (1, \infty)} \frac{(\log u)^{\beta-1}}{u},$$

and for $u > u_0$ the function $\frac{(\log u)^{\beta-1}}{u}$ is decreasing. We write

$$I_{s,r} = \left(\int_{\substack{r < |y| < s \\ |y| \leq 2u_0}} + \int_{\substack{r < |y| < s \\ |y| > 2u_0}} \right) f(s - |y|/2) f(|y|) dy =: I_{s,r}^{(1)} + I_{s,r}^{(2)}$$

(with the convention that the integral over an empty set is equal to zero). When $s > u_0 + 1$, then $I_{s,r}^{(1)} \leq c_5 f(s - u_0) \int_{|y|>r} f(|y|) dy \leq c_6 f(s) \Psi(1/r)$, by the fact that $f(s - u_0) \leq c_7 f(s)$ for some c_7 uniform in s , and by (5.14). On the other hand, when $s \leq u_0 + 1$, then f is within doubling range and we simply have $I_{s,r}^{(1)} \leq f(s/2) \int_{|y|>r} f(|y|) dy \leq c_8 f(s) \Psi(1/r)$. To estimate $I_{s,r}^{(2)}$, we make the following observation: when $u_0 < u < \frac{s}{2}$, then for $\vartheta \in (0, 1)$ one has $s - \vartheta u > u \geq u_0$. Consequently, by the Lagrange's theorem and (5.28), for some $\vartheta \in (0, 1)$,

$$(\log s)^\beta - (\log(s - u))^\beta = \beta \frac{(\log(s - \vartheta u))^{\beta-1}}{s - \vartheta u} u \leq \beta (\log u)^{\beta-1}$$

and further

$$(\log s)^\beta + \frac{1}{2} (\log u)^\beta \leq (\log(s - u))^\beta + (\log(2u))^\beta,$$

increasing u_0 if necessary. This gives that $f(s - u) f(2u) \leq f(s) \exp(-(\theta/2)(\log(u))^\beta)$, for the same range of s and u . Using these observations for y in the domain of $I_{s,r}^{(2)}$ (i.e. $u := |y|/2$), we get

$$I_{s,r}^{(2)} \leq f(s) \int_{|y|>r \vee 2u_0} \exp(-(\theta/2)(\log(|y|/2))^\beta) dy.$$

Since we can directly check that the last integral is dominated by $c_9 \Psi(1/r)$, for every $r > 0$, the claimed bound follows. This completes the proof of the assumption (2) of the cited theorem.

It suffices to prove the remaining condition (3). By [21, Lemma 5 (a)], we have $\psi(x) \asymp \Psi(|x|)$, $x \in \mathbb{R}^d$. Since $\Psi(r) \asymp r^\delta \wedge r^2$ by (5.14), similarly as in the proof of Lemma 5.2 we can show that $\int_{\mathbb{R}^d} e^{-t\psi(z)} |z| dz \leq c_{11} t^{-(d+1)/2}$, for large t . This is exactly the missing assumption (3).

Thus, by [20, Theorem 1] we get

$$p(t, x) \leq c_{12} t f(|x|/4) + c_{13} t^{-d/2} e^{-c_{14} \frac{|x|}{\sqrt{t}} \log(1 + c_{14} \frac{|x|}{\sqrt{t}})}, \quad x \in \mathbb{R}^d, t \geq t_1,$$

for some constants $c_{12} - c_{14}$ and sufficiently large $t_1 > 0$. When $|x| \geq t$, the last exponential member is smaller than $c_{15} f(|x|/4)$, for some constant $c_{15} > 0$. This yields the claimed upper bound for the densities. \square

We now pass to the case when the decay of the Lévy density is stretched exponential, exponential, or superexponential.

Theorem 5.5. *Let X be a symmetric Lévy process with characteristic exponent ψ as in (1.2) with the Gaussian coefficient $A = (a_{ij})_{1 \leq i, j \leq d}$ such that either $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A \xi > 0$, and a symmetric Lévy measure $\nu(dx) = \nu(x)dx$ such that there exist $\theta > 0$, $\beta \in (0, \infty)$, $\gamma \geq 0$ and $\delta \in (0, 2)$ such that either*

$$(5.29) \quad \nu(x) \asymp \mathbf{1}_{\{|x| \leq 1\}} |x|^{-d-\delta} + \mathbf{1}_{\{|x| > 1\}} e^{-\theta(|x|-1)^\beta} |x|^{-\gamma}, \quad x \in \mathbb{R}^d \setminus \{0\},$$

or

$$(5.30) \quad \nu(x) \asymp \mathbf{1}_{\{|x| \leq 1\}} |x|^{-d-\delta}, \quad x \in \mathbb{R}^d \setminus \{0\},$$

(this corresponds to the limiting case $\beta = \infty$). Let V^ω be a Poissonian potential with bounded, compactly supported, nonnegative and nonidentically zero a.e. profile W . Then, for any fixed $x \in \mathbb{R}^d$, one has

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}} \leq - \left(\frac{\rho(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \lambda_{(2)}, \quad \mathbb{Q} - a.s.,$$

and

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}} \geq - \left(\frac{\rho \omega_d(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \lambda_1^{(2)}(B(0, 1)), \quad \mathbb{Q} - a.s.,$$

where $\lambda_1^{(2)}(U)$ and $\lambda_{(2)}$ correspond to the diffusion process with Gaussian matrix \tilde{A} as in (5.20).

In particular, if $A = a \text{Id}$ for some $a \geq 0$ and ν is radial nonincreasing, then

$$\lim_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{d/2}} = - \left(\frac{\rho \omega_d(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \left(a + \frac{1}{2} \int_{\mathbb{R}^d} y_1^2 \nu(y) dy \right) \lambda_1^{BM}(B(0, 1)), \quad \mathbb{Q} - a.s.,$$

where $\lambda_1^{BM}(B(0, 1))$ is the principal eigenvalue of the Brownian motion killed on leaving the ball $B(0, 1)$.

Proof. We proceed along the same scheme as in the proof of Theorem 5.4 above. By Proposition 5.2 the basic assumption **(C)** is satisfied (with \tilde{A} as in (5.20)).

THE UPPER BOUND. We first verify the assumption **(U)**. When $A \equiv 0$, then we derive from the upper bounds in [7, (1.14), (1.17) and (1.21)] that there exist $c_1, c_2 > 0$ such that

$$(5.31) \quad p(t, x) \leq c_1 e^{-c_2 |x|^{(\beta \wedge 1)}}, \quad \text{whenever } |x| \geq 2t \geq 2.$$

This gives that **(U)** is satisfied with $F(r) = e^{-c_3 r^{(\beta \wedge 1)}}$, for some $c_3 \leq c_2$. By Proposition 5.3, this also extends to the case $\inf_{|\xi|=1} \xi \cdot A \xi > 0$ (one may need to adjust constants). On the other hand, Theorem 5.1 yields

$$\lim_{\lambda \rightarrow 0} \lambda^{d/2} \log N^D(\lambda) = -\rho(\lambda_{(2)})^{d/2},$$

where $\lambda_{(2)}$ is determined by the variational formula (5.3) with $\lambda_1^{(2)}(U)$ corresponding to the diffusion process with exponent $\psi^{(2)}(\xi) = \xi \cdot \tilde{A} \xi$, where \tilde{A} is given by (5.20).

We are now ready to apply our general Theorem 3.1 and Corollary 3.1. Recall that $\kappa_0 = \rho(\lambda_{(2)})^{d/2}$ and observe that

$$h_{F, 2, \kappa_0}(t) \approx c_4 \left(\frac{t}{(\log t)^{2/d}} \right)^{1/(\beta \wedge 1)},$$

for some $c_4 > 0$, which implies

$$(5.32) \quad g(t) = \frac{t}{(\log h_{F, 2, \kappa_0}(t))^{2/d}} \approx \frac{(\beta \wedge 1)^{2/d} t}{(\log t)^{2/d}}.$$

Since $Q_1 = \infty$ in Corollary 3.1, we may conclude that

$$\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}} \leq - \left(\frac{\rho(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \lambda_{(2)}, \quad \mathbb{Q} - \text{a.s.},$$

which is the claimed upper bound.

THE LOWER BOUND. Again, by Proposition 5.1, the assumptions of our general Theorem 4.1 and Corollary 4.1 hold with any $K > 1$. Similarly as in the previous proofs, we consider the profile function $F(r) = e^{-c_3 r^{(\beta \wedge 1)}}$ and arbitrary $\kappa > 0$. Moreover, Proposition 5.4 gives that there exists $c_5 > 0$ such that $G(1, R) \geq c_5 e^{-c_3 (R/2)^\beta}$, for large R . This lower estimate is sufficiently sharp for $\beta \in (0, 1]$. However, for $\beta > 1$ it is not sharp enough for our applications. Therefore, we have to address this case separately. According to the definition of the parameter function G in (4.4), we derive from [24, Propositions 3.5 and 3.6] that there exist $c_6, c_7 > 0$ such that for sufficiently large R

$$G(1, R) \geq c_6 e^{-c_7 (\frac{R}{2}) (\log(\frac{R}{2}))^{\frac{\beta-1}{\beta}}}, \quad \text{whenever } \beta \in (1, \infty),$$

and

$$G(1, R) \geq c_6 e^{-c_7 (\frac{R}{2}) \log(\frac{R}{2})}, \quad \text{in the limiting case } \beta = \infty.$$

We are now in a position to apply Corollary 4.1 to our main Theorem 4.1. Observe that $Q_1 = \infty$ in (4.11). By using the above lower bounds for G , we also directly get

$$\lim_{r \rightarrow \infty} \frac{\left| \log G \left(1, \frac{2\sqrt{d}r}{(\log r)^{\frac{2}{d}+2}} \right) \right|}{r \wedge |\log F(r)| + (d/2) \log r} = 0,$$

i.e. one has $Q_2 = 0$ in (4.12). Moreover, note that the asymptotic profile $g(t)$ appearing in (4.14) is κ -independent (cf. (5.32)). Thus, for any fixed $\kappa > 0$ and $x \in \mathbb{R}^d$

$$\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}} \geq - \left(\frac{\rho \omega_d(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \lambda_1^{(2)}(B(0, 1)), \quad \mathbb{Q} - \text{a.s.},$$

which is the required lower bound.

THE CONCLUDING STEP. If $A = a \text{Id}$ for some $a \geq 0$ and ν is radial nonincreasing, then one can show that

$$\lambda_{(2)} = \omega_d^{2/d} \lambda_1^{(2)}(B(0, 1)) = \omega_d^{2/d} \left(a + \frac{1}{2} \int_{\mathbb{R}^d} y_1^2 \nu(y) dy \right) \lambda_1^{BM}(B(0, 1)).$$

Therefore, in this case we have

$$\lim_{t \rightarrow \infty} \frac{u^\omega(t, x)}{t/(\log t)^{d/2}} = - \left(\frac{\rho \omega_d(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \left(a + \frac{1}{2} \int_{\mathbb{R}^d} y_1^2 \nu(y) dy \right) \lambda_1^{BM}(B(0, 1)), \quad \mathbb{Q} - \text{a.s.}$$

The proof is complete. \square

We now illustrate the above result with several important examples.

Example 5.4. (Absolutely continuous Lévy measures with second moment finite). Our Theorem 5.5 above immediately applies to the following examples.

- (1) *Relativistic α -stable process.* When $\psi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ with $\alpha \in (0, 2)$ and $m > 0$, then we have

$$(5.33) \quad \psi(\xi) = \frac{\alpha}{2} m^{1-\frac{2}{\alpha}} |\xi|^2 + o(|\xi|^2) \quad \text{as } |\xi| \rightarrow 0$$

and (5.29) holds with $\theta = m^{1/\alpha}$, $\beta = 1$, $\gamma = (d + 1 + \alpha)/2$ and $\delta = \alpha$. In this case, the quenched behaviour is similar to that for the Brownian motion and we obtain precise first term asymptotics:

$$(5.34) \quad \log u^\omega(t, x) = \frac{\alpha}{2} m^{1-\frac{2}{\alpha}} \lambda_1^{BM}(B(0, 1)) \left(\frac{\rho\omega_d}{d} \right)^{\frac{d}{2}} \frac{t}{(\log t)^{\frac{d}{2}}} + o\left(\frac{t}{(\log t)^{\frac{d}{2}}} \right), \quad \text{as } t \rightarrow \infty.$$

It is instructive to discuss the following two limiting behaviours of the constant appearing in (5.34). When $\alpha \rightarrow 2$, then it tends to $\lambda_1^{BM}(B(0, 1)) \left(\frac{\rho\omega_d}{d} \right)^{\frac{d}{2}}$, which is the constant obtained for the Brownian motion. The second limit is interesting from the mathematical physics point of view. Recall that the Hamiltonian $\sqrt{-\hbar c^2 \Delta + m^2 c^4}$ (called the *Klein-Gordon square root operator* or the *quasi-relativistic Hamiltonian*) is often said to describe the motion of a *free quasi-relativistic particle*. Here m is the mass of a particle, c is the speed of light, and \hbar is the reduced Planck constant. Since the term mc^2 represents the rest mass, the related operator $-L := \sqrt{-\hbar c^2 \Delta + m^2 c^4} - mc^2$ is often called the *kinetic energy operator* (the pure jump Lévy process generated by L is called the *relativistic process* and is determined by its Fourier symbol $\psi(\xi) = \sqrt{\hbar c^2 |\xi|^2 + m^2 c^4} - mc^2$). Observe that in this case (5.33) reads as follows:

$$\psi(\xi) = \frac{\hbar}{2m} |\xi|^2 + o(|\xi|^2) \quad \text{as } |\xi| \rightarrow 0.$$

The leading term is c -independent and it corresponds to passing to the so-called *non-relativistic limit* (i.e. $c \rightarrow \infty$). By this fact also the corresponding leading term in (5.34) remains unchanged under taking such a limit (cf. [17, Remark 1.3]).

- (2) *Isotropic tempered α -stable process.* Let $\nu(x) = C_{d,\alpha} |x|^{-d-\alpha} e^{-m|x|^\beta}$ with $\alpha \in (0, 2)$, $\beta > 0$ and $m > 0$, for some $C_{d,\alpha} > 0$. In this case, one should take $\theta = m$, $\beta > 0$, $\gamma = d + \alpha$ and $\delta = \alpha$ in (5.29). In particular,

$$\psi(\xi) = \left(\frac{C_{d,\alpha}}{2} \int_{\mathbb{R}^d} y_1^2 |y|^{-d-\alpha} e^{-m|y|^\beta} dy \right) |\xi|^2 + o(|\xi|^2) \quad \text{as } |\xi| \rightarrow 0.$$

With this in mind,

$$\begin{aligned} \log u^\omega(t, x) &= \lambda_1^{BM}(B(0, 1)) \left(\frac{C_{d,\alpha}}{2} \int_{\mathbb{R}^d} y_1^2 |y|^{-d-\alpha} e^{-m|y|^\beta} dy \right) \left(\frac{\rho\omega_d(\beta \wedge 1)}{d} \right)^{\frac{2}{d}} \frac{t}{(\log t)^{\frac{d}{2}}} \\ &\quad + o\left(\frac{t}{(\log t)^{\frac{d}{2}}} \right), \quad \text{as } t \rightarrow \infty. \end{aligned}$$

- (3) *Isotropic Lamperti stable process.* Let $\nu(x) = C_{d,\alpha} |x|^{-(d-1)} e^{m|x|} (e^{|x|} + 1)^{-\alpha-1}$ with $\alpha \in (0, 2)$ and $0 < m < \alpha + 1$, for some $C_{d,\alpha} > 0$. For this case we immediately obtain the analogous first term asymptotics as in (2).
- (4) *Truncated stable process.* Let $\nu(x) = C_{d,\alpha} |x|^{-d-\alpha} \mathbf{1}_{\{|x| \leq 1\}}$ with $\alpha \in (0, 2)$, for some $C_{d,\alpha} > 0$. This is the limiting case $\beta = \infty$. We now have

$$\begin{aligned} \log u^\omega(t, x) &= \lambda_1^{BM}(B(0, 1)) \left(\frac{C_{d,\alpha}}{2} \int_{|y| \leq 1} y_1^2 |y|^{-d-\alpha} dy \right) \left(\frac{\rho\omega_d}{d} \right)^{\frac{2}{d}} \frac{t}{(\log t)^{\frac{d}{2}}} + o\left(\frac{t}{(\log t)^{\frac{d}{2}}} \right), \\ &\quad \text{as } t \rightarrow \infty. \end{aligned}$$

As mentioned above, for more clarity we decided to present and prove our Theorems 5.3-5.5 for absolutely continuous Lévy measures only. However, we want to emphasize that similar results holds true in much more general settings. For completeness, we now give some examples of less regular Lévy measures to which our general Theorems 3.1 and 4.1 apply directly. This can be justified by modification of the argument above. The details are left to the reader.

Example 5.5. (Less regular Lévy measures with second moment finite)

(1) *Product Lévy measures.* Let n be a symmetric finite measure on the unit sphere S^{d-1} such that

$$n(B(\varphi, r) \cap S^{d-1}) \geq c_0 r^{d-1}, \quad \varphi \in S^{d-1}, \quad r \in (0, 1/2],$$

for some constant $c_0 > 0$, and let

$$f(s) := \mathbf{1}_{[0,1]}(s) \cdot s^{-\theta/q} + e^m \mathbf{1}_{(1,\infty)}(s) \cdot e^{-ms^\beta} s^{-\delta}, \quad s > 0,$$

with $m > 0$, $\beta \in (0, 1/2]$, $\delta > 0$ and $\theta \in (0, 2q)$. Consider a symmetric Lévy process with Lévy-Khinchine exponent ψ as in (1.2) with diffusion matrix A such that $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A \xi > 0$ and product Lévy measure $\nu(d\varphi) = n(d\varphi)f(r)dr$. Then the \mathbb{Q} -a.s. bounds for $\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}}$ and $\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t/(\log t)^{2/d}}$ of Theorem 5.5 extend to this case. Note that we do not impose any growth condition on n from above. Therefore this example covers a wide range of Lévy measures that are not absolutely continuous with respect to the Lebesgue measure. Many other examples can also be produced by changing the profile f .

(2) *Lévy measures with purely discrete long jumps parts.* Let $f : (-1, 1)^d \cup \mathbb{Z}^d \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{2^{(d+\theta)n}} & \text{when } x \in (-\frac{1}{2^n}, \frac{1}{2^n})^d \setminus [-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}]^d, \quad n \in \mathbb{Z}, \\ \frac{1}{(\max_{1 \leq i \leq d} |x_i|)^{d+\delta}} & \text{when } x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d \setminus \{0\}, \end{cases}$$

with $\theta \in (0, 2)$ and $\delta > d$. Denote by $f_r(x) = f(rx)$, $r > 0$, the dilatations of f . Consider a symmetric Lévy process with Lévy-Khinchine exponent ψ as in (1.2) with diffusion matrix A such that $A \equiv 0$ or $\inf_{|\xi|=1} \xi \cdot A \xi > 0$ and Lévy measure ν_r defined by

$$\nu_r(B) := \int_{B \cap (-1, 1)^d} f_r(y) dy + \sum_{y \in B \cap \mathbb{Z}^d \setminus \{0\}} f_r(y),$$

for every Borel set $B \subset \mathbb{R}^d$ and for given $r > 0$. Then the \mathbb{Q} -a.s. bounds for $\liminf_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{d/(d+2)}}$ and $\limsup_{t \rightarrow \infty} \frac{\log u^\omega(t, x)}{t^{d/(d+2)}}$ with $\delta_1 = \delta$ and $\delta_2 = \delta - d$ as in Theorem 5.3 also apply.

References

- [1] D. Applebaum: *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2nd ed., 2008
- [2] K. Bogdan, T. Grzywny, M. Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. 266 (6) (2014), 3543–3571.
- [3] K. Bogdan et al, *Potential Analysis of Stable Processes and its Extensions* (ed. P. Graczyk, A. Stós), Lecture Notes in Mathematics 1980, Springer, Berlin, 2009.
- [4] L. Brasco, E. Lindgren, E. Parini, *The fractional Cheeger problem*. Interfaces Free Bound. 16 (2014), 419–458. doi: 10.4171/IFB/325
- [5] R. Carmona, J. Lacroix, *Spectral theory of random Schrödinger operators*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1990.
- [6] R. Carmona, W.C. Masters, B. Simon, *Relativistic Schrödinger operators: asymptotic behaviour of the eigenfunctions*, J. Funct. Anal. 91, 1990, 117–142.
- [7] Z.-Q. Chen, P. Kim, T. Kumagai, *Global heat kernel estimates for symmetric jump processes*, Trans. Amer. Math. Soc. 363 (9), 2011, 5021–5055.
- [8] Z.-Q. Chen, R. Song, *Two sided eigenvalue estimates for subordinate processes in domains*, J. Funct. Anal. 226, 2005, 90–113.
- [9] M. Demuth, J. A. van Casteren, *Stochastic Spectral Theory for Self-adjoint Feller Operators. A Functional Analysis Approach*, Birkhäuser, Basel, 2000.
- [10] M.D. Donsker, S.R.S. Varadhan, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math. 28 (1975), no. 4, 525–565.
- [11] R. Fukushima, *From the Lifshitz tail to the quenched survival asymptotics in the trapping problem*, Electron. Commun. Probab. 14 (2009), 435–446.
- [12] T. Grzywny, *On Harnack Inequality and Hölder Regularity for Isotropic Unimodal Lévy Processes*, Potential Anal. 41, 1–29 (2014).
- [13] T. Grzywny, K. Szczypkowski, *Kato classes for Lévy processes*, preprint, 2015, available at arXiv:1503.05747.

- [14] N. Ikeda, S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2-1, 1961, 79-95,
- [15] N. Jacob, *Pseudo-Differential Operators and Markov Processes: Markov Processes and Applications*, vols. 1-3, Imperial College Press, 2003-2005.
- [16] N. Jacob, R. L. Schilling, *Lévy-type processes and pseudo differential operators*. In: O. Barndorff-Nielsen, T. Mikosch and S. Resnick (eds.): *Lévy processes: theory and applications*, Birkhäuser, Boston 2001, 139 -167.
- [17] K. Kaleta, M. Kwaśnicki, J. Małecki, *One-dimensional quasi-relativistic particle in the box*, Rev. Math. Phys. 25, 1350014, 2013.
- [18] K. Kaleta, J. Łórcinzi, *Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes*, Ann. Probab. 43 (3), 2015, 1350-1398.
- [19] K. Kaleta, J. Łórcinzi, *Fall-off of eigenfunctions for non-local Schrödinger operators with decaying potentials*, preprint 2015, available at arXiv:1503.03508.
- [20] K. Kaleta, P. Sztonyk, *Estimates of transition densities and their derivatives for jump Levy processes*, J. Math. Anal. Appl. 431 (1), 2015, 260-282.
- [21] K. Kaleta, P. Sztonyk, *Small time sharp bounds for kernels of convolution semigroups*, J. Anal. Math., to appear, available at arXiv:1403.0912.
- [22] K. Kaleta, K. Pietruska-Pałuba, *Integrated density of states for Poisson-Schrödinger processes on the Sierpiński gasket*, Stochastic Process. Appl. 125 (4), 2015, 1244-1281.
- [23] K. Kaleta, K. Pietruska-Pałuba, *Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket*, preprint 2014, available at arXiv:1410.1715.
- [24] K.-Y. Kim, P. Kim, *Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets*, Stochastic Process. Appl. 124 (9), 2014, 3055-3083.
- [25] W. König, *The parabolic Anderson model*, preprint (2015), available at <http://www.wias-berlin.de/people/koenig/www/PAMsurveyBook.pdf>
- [26] H. Okura, *On the spectral distributions of certain integro-differential operators with random potential*, Osaka J. Math. 16 (1979), no. 3, 633-666.
- [27] H. Okura, *Some limit theorems of Donsker-Varadhan type for Markov processes expectations*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 57 (1981), 419-440.
- [28] K. Pietruska-Pałuba, *The Lifschitz singularity for the density of states on the Sierpinski gasket*, Probab. Theory Related Fields 89 (1991), no. 1, 1-33.
- [29] K. Pietruska-Pałuba, *Almost sure behaviour of the perturbed Brownian motion on the Sierpinski gasket*. Stochastic Process. Appl. 85 (2000), no. 1, 1-17.
- [30] W.E. Pruitt, *The growth of random walks and Lévy processes*, Ann. Probab. 9 no. 6, 948-956 (1981).
- [31] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 1999.
- [32] A.S. Sznitman, *Brownian asymptotics in a Poissonian environment*. Probab. Theory Relat. Fields 95 (1993), 155-174.
- [33] R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions*, Walter de Gruyter, 2010.
- [34] R. Schilling, *Growth and Hölder conditions for the sample paths of Feller processes*, Probab. Theory Relat. Fields 112, 565-611 (1998).
- [35] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics 265, Springer 2012.

K. KALETA, INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSZAWA AND FACULTY OF PURE AND APPLIED MATHEMATICS, WROCŁAW UNIVERSITY OF TECHNOLOGY, WYB. WYSPIAŃSKIEGO 27, 50-370 WROCŁAW, POLAND

E-mail address: kamil.kaleta@pwr.edu.pl, kkaleta@mimuw.edu.pl

K. PIETRUSKA-PAŁUBA, INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSZAWA, POLAND

E-mail address: kpp@mimuw.edu.pl